11

X-520-62-222

TM X-55048

PLANS OFFICE TECHNICAL REPORT NO. 7

A GENERAL LEAST SQUARES FORMULATION FOR TRACKING SYSTEM ACCURACY ANALYSIS — WITH APPLICATION TO THE WOOMERA STATION DATA OF MA-6 MISSION

PREPARED BY

HOWARD H. BROWN

NOVEMBER 1, 1962

GODDARD SPACE FLIGHT CENTER

GREENBELT, MD.

PLANS OFFICE TECHNICAL REPORT NO. 7

A General Least Squares Formulation for Tracking System Accuracy Analysis - With Application to the Woomera Station Data of MA-6 Mission

by

Howard H. Brown

I. Summary

A general least squares procedure has been formulated for the analysis of a tracking system's accuracy based on examination of the range, azimuth, and elevation outputs of the tracking system. The least squares process for calculating the standard deviations of range errors, azimuth errors, and elevation errors is presented in Part II.

At the time at which this study was initiated, the data of the Woomera FPS-16 station was of particular interest because the burning of the last powered flight stage of the Centaur mission terminated in a region over this station. The accuracy of measured values of range, azimuth, and elevation as well as time derivatives of these variables were of natural interest for the injection problem. For these reasons the data taken on the MA-6 mission

at the Woomera station has been used to illustrate the application of the derived least squares method.

Typical standard deviations of range errors, azimuth errors, and elevation errors are presented in Figures 1, 2, and 3. These figures show that computed standard deviations are strongly dependent on the degree of least squares fit of range, azimuth, or elevation. In general, only fourth or fifth degree polynomials produced suitable fittings for these variables, based on the criterion that the computed standard deviations of the errors changed little in going from fourth to fifth degree least squares fit. The strong dependence of the standard deviations σ_r , σ_α , and σ_ε on the shapes of range, azimuth, and elevation vs. time curves is indicated in Figure 4.

It must be expected that some mission flights will yield radar data requiring for analysis higher degree least squares polynomials than those used in this study.

Part III develops a process for calculating the standard deviations of range rate errors, azimuth rate errors, and elevation rate errors for a system in which range rate, azimuth rate, and elevation rate are computed from the time derivatives of the corresponding least squares polynomials used to fit range, azimuth, and elevation. Since the process involves considerable use of the concepts of variance, covariance, and expectation of random variables, a review of the necessary concepts and operations is given in Part III, subheading 1. Average standard deviations of range rate errors, azimuth rate errors, and elevation rate errors obtained by these methods are presented in Figures 10, 11, and 12. The variations of the various standard deviations $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, and $\sigma_{\dot{\varepsilon}}$ with time over a one minute interval are shown in Figures 5, 6, 7, 8, and 9.

The results in Figures 10, 11, and 12 indicate that quite reasonable values of $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, and $\sigma_{\dot{\varepsilon}}$ are obtained when the corresponding least squares polynomials for range, azimuth, and elevation are of the fourth or fifth degree.

The methods of Part III for calculating $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, and $\sigma_{\dot{\varepsilon}}$ apply only when the rate variables $\dot{r}$, $\dot{\alpha}$, and $\dot{\varepsilon}$ are produced as indicated in the text.

All of the calculations of Part III are based on taking $\dot{r}$, $\dot{\alpha}$, and $\dot{\varepsilon}$ from least squares fitting over a one minute interval with ten available samples of r, α , and ε . Because of the wide variations of $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, and $\sigma_{\dot{\varepsilon}}$ with time as exhibited in Figures 5, 6, 7, 8, and 9, it is natural to ask whether these variations could be limited by altering the number of samples used to make the least squares fits. This question is answered in the affirmative in Part IV, which develops in an approximate manner the dependence of σ_r, σ_α, σ_ε, $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, and $\sigma_{\dot{\varepsilon}}$ on the number (N) of samples used to make least squares polynomial fits for r, α, and ε. The results, shown in Figures 13 and 14 indicate that by the use of a sufficiently large sample size N, σ_r, σ_α, and σ_ε will have constant values over the least squares fitting interval, while the corresponding standard deviations of the rate errors $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, and $\sigma_{\dot{\varepsilon}}$ may be reduced to extremely small values.

Consideration of the use of the methods of Part III to produce operational values of $\dot{r}$, $\dot{\alpha}$, and ε must take account of the time lag from a measured range, azimuth, or elevation to the corresponding $\dot{r}$, $\dot{\alpha}$, and $\dot{\varepsilon}$ obtained from the least squares fitting.

In the least squares procedures of this study no attempt has been made to compute cross correlation among measured tracking variables r, α, and ε, although such correlations exist.

Contents

Notation Conventions

Measured values of a variable are distinguished from correct or theoretical values by a star on the measured quantity. E.g., r^* is a measured value of range, while r is the theoretical value of the same variable.

Letter symbols which denote matrices are distinguished from ordinary variables by underlining the matrix symbol. Further, upper case letters denote square matrices; lower case letters denote column (or row) matrices.

E.g. $\underset{(5x5)}{\underline{K}}$ denotes a 5x5 square matrix

$\underset{(5x1)}{\underline{b}}$ denotes a column matrix with five rows.

Matrices with all elements indicated are enclosed with brackets, and are not underlined. Since no vectors are used in this study, the underlined symbols will always be understood as matrices.

In least squares polynomials involving time, t, as independent variable, the unit of time is a time interval of six seconds or one tenth minute duration. Thus t=1 corresponds to six seconds, t=2 corresponds to twelve seconds, etc.

II. The Least Squares Principle

To illustrate the least squares method we shall assume that over a one minute tracking duration the theoretical range is given in terms of time by a fourth degree polynomial

$$r = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 \qquad (1)$$

Available for the determination of the coefficients b_0, b_1, b_2, b_3, b_4 are ten measured values of range $r_1^*, r_2^*, \ldots, r_{10}^*$ recorded at equi-spaced values of time, six seconds apart. It is convenient to denote these time values by: $t_1, 2t_1, 3t_1, \ldots, 10t_1$. An observed value of range, r_j^*, will differ from the corresponding theoretical range determined from equation (1) by a quantity $\Delta r_j = r_j^* - r_j$, which we call the deviation of the observed range from the theoretical range. Throughout this study a starred value of a variable, e.g. r*, will denote a measured value of that variable; the corresponding symbol without star will denote a theoretical value of the variable.

For $j = 1, 2, \ldots, n$ the squares of the deviations may be written, from equation (1) and the definition of Δr :

$$(\Delta r_1)^2 = (b_0 + b_1 t + b_2 t_1^2 + b_3 t_1^3 + b_4 t_1^4 - r_1^*)^2 \qquad (2a)$$

$$(\Delta r_2)^2 = (b_0 + 2 b_1 t_1 + 2^2 b_2 t_1^2 + 2^3 b_3 t_1^3 + 2^4 b_4 t_1^4 - r_2^*)^2 \qquad (2b)$$

$$(\Delta r_3)^2 = (b_0 + 3 b_1 t_1 + 3^2 b_2 t_1^2 + 3^3 b_3 t_1^3 + 3^4 b_4 t_1^4 - r_3^*)^2 \qquad (2c)$$

$$\vdots \qquad \qquad \vdots$$

$$(\Delta r_n)^2 = (b_0 + n b_1 t_1 + n^2 b_2 t_1^2 + n^3 b_3 t_1^3 + n^4 b_4 t_1^4 - r_n^*)^2 \qquad (2n)$$

Form the sum of the squares of the deviations

$$S = \sum_{j=1}^{n} (\Delta r_j)^2 \qquad (3)$$

The function S will always be a homogeneous quadratic form in the coefficient variables b_0, b_1, ..., b_4.

The measured values of r: r_1^*, r_2^*, ... , r_n^* are recorded at measured values of time. The possible errors in measured times versus the theoretical values t_1, $2t_1$, $3t_1$, ... , nt_1, are considered quite insignificant compared to the errors in measured ranges.

The least squares principle[1] asserts that the set of coefficients b_0, b_1, b_2, b_3, b_4 is best for which the sum S of the squares of the deviations is a minimum. By a fundamental theorem of calculus a minimum of S will be obtained when the partial derivatives of S with respect to b_0, b_1, b_2, b_3, and b_4 are all simultaneously zero.

The required partial derivatives are readily obtained from the forms (2a), (2b),......, (2n) of the individual squared deviations.

---

[1] Rererence (2), pp. 288-291; reference (8), pp. 242-255; reference (9), pp. 414-424, and 466-470. Each of these references describes the least squares principle. The complete proof that the procedure outlined here yields a minimum of S is given in reference (8).

Thus

$$\begin{aligned}\frac{\partial S}{\partial b_0} = 2\Big\{ & \left(b_0 + b_1t_1 + b_2t_1^2 + b_3t_1^3 + b_4t_1^4 - r_1^*\right) \\ & + \left(b_0 + 2b_1t_1 + 2^2b_2t_1^2 + 2^3b_3t_1^3 + 2^4b_4t_1^4 - r_2^*\right) \\ & + \ldots\ldots + \left(b_0 + nb_1t_1 + n^2b_2t_1^2 + n^3b_3t_1^3 + n^4b_4t_1^4 - r_n^*\right)\Big\} = 0 \qquad (4a)\end{aligned}$$

$$\begin{aligned}\frac{\partial S}{\partial b_1} = 2\Big\{ & t_1\left(b_0 + b_1t_1 + b_2t_1^2 + b_3t_1^3 + b_4t_1^4 - r_1^*\right) \\ & + 2t_1\left(b_0 + 2b_1t_1 + 2^2b_2t_1^2 + 2^3b_3t_1^3 + 2^4b_4t_1^4 - r_2^*\right) \\ & + \ldots\ldots + nt_1\left(b_0 + nb_1t_1 + n^2b_2t_1^2 + n^3b_3t_1^3 + n^4b_4t_1^4 - r_n^*\right)\Big\} = 0 \qquad (4b)\end{aligned}$$

$$\begin{aligned}\frac{\partial S}{\partial b_2} = 2\Big\{ & t_1^2\left(b_0 + b_1t_1 + b_2t_1^2 + b_3t_1^3 + b_4t_1^4 - r_1^*\right) \\ & + (2t_1)^2\left(b_0 + 2b_1t_1 + 2^2b_2t_1^2 + 2^3b_3t_1^3 + 2^4b_4t_1^4 - r_2^*\right) \\ & + \ldots\ldots + (nt_1)^2\left(b_0 + nb_1t_1 + n^2b_2t_1^2 + n^3b_3t_1^3 + n^4b_4t_1^4 - r_n^*\right)\Big\} = 0 \qquad (4c)\end{aligned}$$

$$\begin{aligned}\frac{\partial S}{\partial b_3} = 2\Big\{ & t_1^3\left(b_0 + b_1t_1 + b_2t_1^2 + b_3t_1^3 + b_4t_1^4 - r_1^*\right) \\ & + (2t_1)^3\left(b_0 + 2b_1t_1 + 2^2b_2t_1^2 + 2^3b_3t_1^3 + 2^4b_4t_1^4 - r_2^*\right) \\ & + \ldots\ldots + (nt_1)^3\left(b_0 + nb_1t_1 + n^2b_2t_1^2 + n^3b_3t_1^3 + n^4b_4t_1^4 - r_n^*\right)\Big\} = 0 \qquad (4d)\end{aligned}$$

$$\begin{aligned}\frac{\partial S}{\partial b_4} = 2\Big\{ & t_1^4\left(b_0 + b_1t_1 + b_2t_1^2 + b_3t_1^3 + b_4t_1^4 - r_1^*\right) \\ & + (2t_1)^4\left(b_0 + 2b_1t_1 + 2^2b_2t_1^2 + 2^3b_3t_1^3 + 2^4b_4t_1^4 - r_2^*\right) \\ & + \ldots\ldots + (nt_1)^4\left(b_0 + nb_1t_1 + n^2b_2t_1^2 + n^3b_3t_1^3 + n^4b_4t_1^4 - r_n^*\right)\Big\} = 0 \qquad (4e)\end{aligned}$$

Equations (4a), (4b), (4c), (4d), (4e) comprise a set of five simultaneous linear equations for the unknown coefficients b_0, b_1, b_2, b_3, b_4. These equations may be systematically rearranged, with the observed quantities r_1^*, r_2^*, . . . , r_n^* on the right, into the matrix form

$$\begin{bmatrix} n & t_1\sum_{j=1}^{n} j & t_1^2\sum_{j=1}^{n} j^2 & t_1^3\sum_{j=1}^{n} j^3 & t_1^4\sum_{j=1}^{n} j^4 \\ t_1\sum_{j=1}^{n} j & t_1^2\sum_{j=1}^{n} j^2 & t_1^3\sum_{j=1}^{n} j^3 & t_1^4\sum_{j=1}^{n} j^4 & t_1^5\sum_{j=1}^{n} j^5 \\ t_1^2\sum_{j=1}^{n} j^2 & t_1^3\sum_{j=1}^{n} j^3 & t_1^4\sum_{j=1}^{n} j^4 & t_1^5\sum_{j=1}^{n} j^5 & t_1^6\sum_{j=1}^{n} j^6 \\ t_1^3\sum_{j=1}^{n} j^3 & t_1^4\sum_{j=1}^{n} j^4 & t_1^5\sum_{j=1}^{n} j^5 & t_1^6\sum_{j=1}^{n} j^6 & t_1^7\sum_{j=1}^{n} j^7 \\ t_1^4\sum_{j=1}^{n} j^4 & t_1^5\sum_{j=1}^{n} j^5 & t_1^6\sum_{j=1}^{n} j^6 & t_1^7\sum_{j=1}^{n} j^7 & t_1^8\sum_{j=1}^{n} j^8 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} \tag{5}$$

where: $W_0 = \sum_{j=1}^{n} r_j^*$, $W_1 = t_1 \sum_{j=1}^{n} j r_j^*$, $W_2 = t_1^2 \sum_{j=1}^{n} j^2 r_j^*$, $W_3 = t_1^3 \sum_{j=1}^{n} j^3 r_j^*$

and $W_4 = t_1^4 \sum_{j=1}^{n} j^4 r_j^*$

The unit time interval t_1, in equation (5) may be any convenient value. If the variable r is to be fitted over a time duration T, then $t_1 = T/n$. For the data to be examined in this study T=60 seconds and n=10. It is convenient to take t_1 as a time unit and to set t_1 equal to unity throughout equation (5). Hereafter all calculations in this study will be made on this basis. With this simplification, the unknown least squares coefficients are obtained by inverting the matrix equation (5). Writing (5) in the abbreviated matrix form

$$\underset{(5x5)}{\underline{K}} \; \underset{(5x1)}{\underline{b}} = \underset{(5x1)}{\underline{w}} \tag{6}$$

$$\underset{(5x1)}{\underline{b}} = \underset{(5x5)}{\underline{K}^{-1}} \; \underset{(5x1)}{\underline{w}} \tag{7}$$

where $\underline{w}$ is the column matrix on the right side of equation (5), $\underline{b}$ is the column matrix of least squares coefficients, and $\underline{K}$ is the 5x5 matrix in (5) with $t_1 = 1$.

Analogous to equation (5), we may assume for the azimuth angle a least squares polynomial fit of the form

$$\alpha = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \tag{8}$$

with available observed values of azimuth α_1^*, α_2^*, ... , α_{10}^* again taken at equi-spaced times t_1, $2t_1$, $3t_1$, ... , $10t_1$. Minimization of the sum of the squares of the deviations of the observed values from the theoretical curve then leads to a matrix equation

$$\underset{(5\times 5)}{\underline{K}}\ \underset{(5\times 1)}{\underline{a}} = \underset{(5\times 1)}{\underline{w}} = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \tag{9}$$

where now $w_0 = \sum_{j=1}^{n} \alpha_j^*$, $w_1 = \sum_{j=1}^{n} j\,\alpha_j^*$,...., etc.
and K is the same numerical 5x5 matrix which appears in equation (5). Again the column matrix of least squares coefficients

$$\underline{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}_{(5\times 1)}$$

is obtained by inverting (9)

so that

$$\underset{(5\mathrm{x}1)}{\underline{a}} = \underset{(5\mathrm{x}5)}{\underline{K}^{-1}} \; \underset{(5\mathrm{x}1)}{\underline{w}} \tag{10}$$

Similarly, a fourth degree polynomial least squares fit for elevation angle in the form

$$\varepsilon = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 \tag{11}$$

over the same time interval $t_1 \leq t \leq 10 t_1$ results when the column matrix of least squares coefficients

$$\underline{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}_{(5\times 1)}$$

is determined by

$$\underset{(5\mathrm{x}1)}{\underline{c}} = \underset{(5\mathrm{x}5)}{\underline{K}^{-1}} \; \underset{(5\mathrm{x}1)}{\underline{w}} \tag{12}$$

where now $W_0 = \sum_{j=1}^{n} \varepsilon_j$, $W_1 = \sum_{j=1}^{n} j\,\varepsilon_j$, $W_2 = \sum_{j=1}^{n} j^2 \varepsilon_j$, ..., etc. Again K^{-1} is the same (5x5) matrix appearing in equations (7) and (10).

The equations of the form (7), (10), (11) are easily generalized to the determination of the least squares coefficients for a least squares polynomial fitting of any degree. In least squares terminology the equations (6), (9), (12) are called the normal equations.

When the least squares coefficients are determined from the appropriate relations (7), (10), and (12) for a given one minute time interval, the resulting polynomials (1), (8), and (11) are the best fourth degree polynomial representations in the least squares sense for the range, azimuth angle, and elevation angle, respectively, for that particular time interval.

From the least squares polynomial representation (1) for range, best theoretical values of range at the data times: t_1, $2t_1$, $3t_1$, ... , $10t_1$ may be computed by inserting t=1, 2,......., 10 in equation (1). Denoting these values of r by r_1, r_2, ... , r_{10}, the substitution process may be condensed into the single matrix formula

$$\begin{bmatrix} r_1 \\ r_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ r_{10} \end{bmatrix}_{(10\times 1)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 128 \\ 1 & 5 & 25 & 125 & 625 \\ 1 & 6 & 36 & 216 & 1296 \\ 1 & 7 & 49 & 343 & 2401 \\ 1 & 8 & 64 & 512 & 4096 \\ 1 & 9 & 81 & 729 & 6561 \\ 1 & 10 & 100 & 1000 & 10000 \end{bmatrix}_{(10\times 5)} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}_{(5\times 1)} \quad (14)$$

When this array of theoretical values of r is subtracted from the corresponding array of observed values there results the column array of residuals or deviations $\underline{V}_r$:

$$\underline{V}_r = \begin{bmatrix} r_1^* - r_1 \\ r_2^* - r_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ r_{10}^* - r_{10} \end{bmatrix}_{(10\times 1)} \tag{15}$$

Note that

$$\underline{V}_r^T \underline{V}_r = (r_1^* - r_1)^2 + (r_2^* - r_2)^2 + \ldots\ldots + (r_n^* - r_n)^2$$
$$= \sum_{j=1}^{n} (r_j^* - r_j)^2$$

where $\underline{V}_r^T$ is the transpose of the column matrix $\underline{V}_r$.

The standard deviation of the range errors may then be computed by (Reference 11, pp. 185-191).

$$\sigma_r = \sqrt{\frac{\underline{V}_r^T \underline{V}_r}{n - (k+1)}} \tag{16}$$

where k is the degree of the least squares polynomial fit, (k+1) is the rank of the set of linear equations (5), and n is the number of observed values of r used to make the least squares fit. Similarly, the standard deviations of azimuth and elevation errors may be computed by

$$\sigma_\alpha = \sqrt{\frac{\underline{V}_\alpha^T \underline{V}_\alpha}{n-(k+1)}} \tag{17}$$

$$\sigma_\varepsilon = \sqrt{\frac{\underline{V}_\varepsilon^T \underline{V}_\varepsilon}{n-(k+1)}} \tag{18}$$

In each of the formulae (16), (17), (18) the number n-(k+1) is the number of degrees of freedom. In this study least squares fittings are made over a one minute interval, with n=10 available samples per minute interval. The formulae (16), (17), (18) are not exact formulae; rather, they are "Asymtotic Estimates" (Reference 11, p. 188) for the standard deviations σ_r, σ_α, and σ_ε, since the sets of recorded values of range, azimuth, and elevation constitute only finite samples taken from infinite populations of the measured variables.

The procedure for determining the coefficients in the least squares polynomial fittings by minimization of the quantity S defined by equation (3) has been stated here for unit weights on the individual $(\Delta r_j)^2$. It is possible to generalize the procedure so that different weights (other than unity) are assigned to the various $(\Delta r_j)^2$. However, such additional smoothing is not considered applicable for this study.

The least squares process outlined here has been applied to the Woomera station FPS-16 radar data recorded in the MA-6 mission. The necessary computations were performed on the GSFC 7090 computer. The least squares fittings of range, azimuth, and elevation were made with all least squares polynomials from degree two through degree five. The standard deviations of range errors, azimuth errors, and elevation errors are summarized in Figures 1, 2, and 3. The results with second order polynomials were quite unsatisfactory and are therefore omitted in these figures. The results in these three figures show that computed values of σ_r, σ_α, σ_ε are strongly dependent on the degree of the least squares polynomials used to fit range, azimuth, and elevation.

Table I is a typical IBM 7090 printout for a fourth degree least squares fit of range, azimuth, and elevation over a one minute tracking interval, together with the computed standard deviations of range, azimuth, and elevation errors.

The results depicted in Figures 1, 2, and 3 are to be regarded as average standard deviations of range errors, azimuth errors, and elevation errors over a one minute tracking interval based on least squares fitting with ten samples per minute. The methods of Part III show that the standard deviations σ_r , σ_α , σ_ε actually vary with time over a one minute interval of least squares fit. Part IV shows that this dependence on time may be reduced by using a larger sample size for the least squares fit.

The standard deviations presented in Figures 1, 2, and 3 are intended to illustrate those errors in range, azimuth, and elevation which may be represented statistically. It should not be inferred, however, that these errors represent the entire errors in range, azimuth, and elevation. A range tracking system will have dynamic steady state system errors of the form (Reference 5)

$$\varepsilon_{ss} = \frac{R_i}{1+K_p} + \frac{\dot{R}_i}{K_v} + \frac{\ddot{R}_i}{K_a} + \cdots + \frac{\overset{(m)}{R_i}}{K_m}$$

where K_p, K_v, K_a,....etc. are gain constants of the tracking loop and R_i , $\dot{R}_i$, $\ddot{R}_i$,....etc. are the input range and its time derivatives. The design of a range tracking system attempts to limit such errors with respect to anticipated forms of the input range $R_i(t)$. Practical limitations require the termination of the expression for ε_{SS} at a definite $R_i^{(m)}$, which involves the assumption that derivatives of $R_i(t)$ beyond the m^{th} are negligible. Unfortunately, this condition may be violated for many range inputs $R_i(t)$ which are high degree polynomials. This seems to be the cause of the unreasonably large values of σ_r, σ_α and σ_ε shown for certain time intervals on Figure 4. Thus, there is always the possibility that a particular range input $R_i(t)$ will result in steady state error ε_{SS} not only larger than the design specification but growing with time. Analogous expressions apply for steady state angle tracking errors.

The strong dependence of σ_r, σ_α, and σ_ε on the shape of the range, azimuth, and elevation vs. time curves is shown in Figure 4. In this figure the standard deviation of azimuth error, σ_α, is omitted for the time period from one to two minutes; the azimuth data for this period varied too radically to be fitted with a fourth degree least squares polynomial.

III. Determination of Standard Deviations of Range Rate, Azimuth Rate, and Elevation Rate from the Least Squares Polynomials for Range, Azimuth, and Elevation

1. Review of Operations with Expection, Variance, and Covariance

The standard deviations of range rate, azimuth rate, and elevation rate may be computed from the corresponding least squares representations for range, azimuth, and elevation on the assumption that the time derivatives of the least squares fits for r, α, ε, are reasonable representations for the theoretical values of $\dot{r}$, $\dot{\alpha}$, and $\dot{\alpha}$. The determination of $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, and $\sigma_{\dot{\varepsilon}}$ by this method requires the application of certain elementary theorems and operations dealing with the expectation, variance, and covariance of random variables. It is pertinent to review the necessary operations, theorems, and definitions here. The material is taken from reference 2.

If X is a random variable, we denote by E(X) the expectation of the random variable. If the first probability density function f(X) of the random variable is known, then

$$E(X) = \int_{-\infty}^{\infty} X f(X)\, dX = \mu_X \tag{19}$$

the mean of X, and

$$E(X^2) = \int_{-\infty}^{\infty} X^2 f(X)\, dX = \sigma_X^2 + \mu_X^2 \tag{20}$$

If X_1, X_2 are random variables then the expectation of the sum $X_1 + X_2$ is (Reference 2, p..165)

$$E(X_1 + X_2) = E(X_1) + E(X_2) \tag{21}$$

The covariance of two random variables X and Y, written cov (X,Y), is defined by (Reference 2, pp. 169-170; reference 3, p. 356)

$$\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y) \tag{22}$$

Since $E(X) = \mu_X$ and $E(Y) = \mu_Y$ the last term in equation (22) is $\mu_X \mu_Y$. In particular

$$\begin{aligned} \mathrm{cov}(X,X) &= E(X^2) - [E(X)]^2 \\ &= \sigma_X^2, \text{ the variance of X} \end{aligned} \tag{23}$$

Computation of the expectation term E(XY) in equation (22) requires in general knowledge of the probability density function of the product XY. In practice this difficulty may be circumvented provided sufficiently simple algebraic relations exist between the two random variables.

It will be useful for the purposes of this study to determine the covariance of two random variables X, Y which are linearly related. Let the linear relation be $Y=aX + b$ where a, b are scalars. Then, by the definition for covariance

$$\begin{aligned} \mathrm{cov}(X,Y) &= E\left(aX^2 + bX\right) - E(X)E(aX+b) \\ &= a\left(\sigma_X^2 + \mu_X^2\right) + b\mu_X - \mu_X(a\mu_X + b) \\ &= a\,\sigma_X^2 \qquad (24) \end{aligned}$$

Thus, if $b_0, b_1, \ldots\ldots, b_K$ are a set of least squares coefficients, it is established in appendix A that any coefficient b_P may be expressed linearly in terms of b_0 in the form

$$b_P = \left(\frac{W_0}{W_P}\right) b_0 + C_P \qquad (25)$$

for $P = 1, 2, \ldots, K$.

where $\frac{W_0}{W_P}$ and C_P are constants. Then, by application of (24)

$$\mathrm{cov}(b_0, b_P) = \left(\frac{W_0}{W_P}\right)\sigma_{b_0}^2 \tag{26}$$

Further

$$\begin{aligned}
\mathrm{cov}(b_i, b_j) &= \mathrm{cov}\left[\left(\frac{W_0}{W_i}\right)b_0 + C_i\,,\,\left(\frac{W_0}{W_j}\right)b_0 + C_j\right] \\
&= E\left[\left(\frac{W_0^2}{W_i W_j}\right)b_0^2 + \left(\frac{W_0 C_i}{W_j} + \frac{W_0 C_j}{W_i}\right)b_0 + C_i C_j\right] \\
&\qquad - E\left[\left(\frac{W_0}{W_i}\right)b_0 + C_i\right] E\left[\left(\frac{W_0}{W_j}\right)b_0 + C_j\right] \\
&= \frac{W_0^2}{W_i W_j}\left(\sigma_{b_0}^2 + \mu_{b_0}^2\right) + \left(\frac{W_0 C_i}{W_j} + \frac{W_0 C_j}{W_i}\right)\mu_{b_0} + C_i C_j \\
&\qquad - \left(\frac{W_0 \mu_{b_0}}{W_i} + C_i\right)\left(\frac{W_0 \mu_{b_0}}{W_j} + C_j\right) \\
&= \left(\frac{W_0^2}{W_i W_j}\right)\sigma_{b_0}^2
\end{aligned} \tag{27}$$

where (i, j) are any two integers in the set $(0, 1, 2, \ldots\ldots\ldots k)$. The constants W_0 , W_i , W_j are defined in terms of the data that is being fitted. For a least squares fit of range W_0 , W_i , and W_j are defined by the formulae

$$W_0 = \sum_{\nu=1}^{n} r_\nu^* \ , \quad W_i = \sum_{\nu=1}^{n} (\nu)^i r_\nu^* \ , \quad W_j = \sum_{\nu=1}^{n} (\nu)^j r_\nu^*$$

where the r_ν^* are measured values of range recorded at equal time intervals.

If (X_0, X_1, ... , X_K) is a set of random variables the complete variance-covariance properties of the collection of random variables may be conveniently and systematically expressed by the covariance matrix

$$C = \begin{bmatrix} cov(X_0,X_0) & cov(X_0,X_1) & \cdots & cov(X_0,X_K) \\ cov(X_1,X_0) & cov(X_1,X_1) & \cdots & cov(X_1,X_K) \\ \vdots & \vdots & & \vdots \\ cov(X_K,X_0) & cov(X_K,X_1) & \cdots & cov(X_K,X_K) \end{bmatrix} \qquad (28)$$

where the diagonal elements are respectively $\sigma^2_{X_0}$, $\sigma^2_{X_1}$, ... , $\sigma^2_{X_K}$. It should be noted that C is always a symmetric matrix since $cov(X_i,X_j) = cov(X_j,X_i)$.

The two expressions (26) and (27) permit computation in terms of σ_{b_0} the complete covariance matrix for a set of least squares coefficients b_0 , b_1 , ... , b_K , considered as random variables[2/]. Thus, when $k=4$ the complete covariance matrix of the random variables b_0, b_1 , . . . , b_4 may be written

---

[2/] To be rigorous, we mean the random variables Δb_0, Δb_1, ... , Δb_K , the deviations of b_0, b_1 , . . . , b_K from their means. However, the variance-covariance properties of Δb_0, Δb_1 . . . . , Δb_K (about zero means) are the same as the variance-covariance properties of b_0, b_1 , ... , b_K . Thus $var(b_j) = var(\Delta b_j)$; $cov(\Delta b_0, \Delta b_j) = cov(b_0, b_j)$; $cov(\Delta b_i, \Delta b_j) = cov(b_i, b_j)$; etc. For this reason, in the statements beginning with equation (26) and hereafter, we omit writing the Δ symbol in such expressions involving the b's .

$$C = \sigma_{b_0}^2 \begin{bmatrix} 1 & \frac{W_0}{W_1} & \frac{W_0}{W_2} & \frac{W_0}{W_3} & \frac{W_0}{W_4} \\ \frac{W_0}{W_1} & \left(\frac{W_0}{W_1}\right)^2 & \frac{W_0^2}{W_1 W_2} & \frac{W_0^2}{W_1 W_3} & \frac{W_0^2}{W_1 W_4} \\ \frac{W_0}{W_2} & \frac{W_0^2}{W_2 W_1} & \left(\frac{W_0}{W_2}\right)^2 & \frac{W_0^2}{W_2 W_3} & \frac{W_0^2}{W_2 W_4} \\ \frac{W_0}{W_3} & \frac{W_0^2}{W_3 W_1} & \frac{W_0^2}{W_3 W_2} & \left(\frac{W_0}{W_3}\right)^2 & \frac{W_0^2}{W_3 W_4} \\ \frac{W_0}{W_4} & \frac{W_0^2}{W_4 W_1} & \frac{W_0^2}{W_4 W_2} & \frac{W_0^2}{W_4 W_3} & \left(\frac{W_0}{W_4}\right)^2 \end{bmatrix} \tag{29}$$

The collection of least squares coefficients is considered as a collection of random variables in the following sense. Consider a time interval for which there is available a large number of data samples of one variable, say range. Let the collection of values of r be divided into sets as follows:

$$
\begin{array}{llllll}
S_1: & r_1^{(1)}, & r_2^{(1)}, & \cdots\cdots\cdots & , & r_n^{(1)} \\
S_2: & r_1^{(2)}, & r_2^{(2)}, & \cdots\cdots\cdots & , & r_n^{(2)} \\
\vdots & \vdots & \vdots & & & \vdots \\
S_\lambda: & r_1^{(\lambda)}, & r_2^{(\lambda)}, & \cdots\cdots\cdots & , & r_n^{(\lambda)}
\end{array}
$$

for a given set S_i , make a least squares fit of r in the form

$$r = b_0^{(i)} + b_1^{(i)} t + \cdots\cdots + b_k^{(i)} t^k$$

For each set of data values S_i there will be a set of least squares coefficients $b_0^{(i)}$, $b_1^{(i)}$,........., $b_k^{(i)}$. Thus there will be λ different values of b_0 , λ different values of b_1 , ..., λ different values of b_k . Hence, each coefficient b_p will have a probability density function $\mathcal{P}(b_p)$, and there may be computed for each b_p a standard deviation σ_{b_p} .

In this study it has been found unnecessary to compute the various standard deviations σ_{b_p} ; we establish, rather, relations (equations 26 and 27) that reduce the problem to the computation of a single sigma, i.e., σ_{b_0} , which is then determined from the data of Part II.

Finally, if X_1 , X_2 ...., X_k are random variables, a fundamental theorem (Reference 2, p. 170; reference 6, pp. 179-180) states that the variance of their sum is

$$Var\,(X_1 + X_2 + \dots + X_k) = Var(X_1) + Var(X_2) + \dots + Var(X_k) + 2\sum_{i,j} cov(X_i, X_j) \qquad (30)$$

where the summation, $\sum_{i,j}$, is over all pairs of integers (i,j) for which $i \neq j$ and $1 \leq i \leq k$, $1 \leq j \leq k$.

The variance of (aX + b) where X is a random variable and a,b are constants is (Reference 6, p. 179)

$$Var(aX+b) = a^2 Var(X) = a^2 \sigma_X^2$$

Thus, if $b_P = \left(\frac{W_o}{W_P}\right) b_o + C_P$ (equation 25) where $\frac{W_o}{W_P}$ and C_P are constants, then

$$Var(b_P) = \sigma_{b_P}^2 = \left(\frac{W}{W_P}\right)\sigma_{b_o}^2 \qquad (31)$$

for $P = 0, 1, \dots, k$.

To illustrate the application of these principles for operations with variance and covariance, if

$$r = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 \tag{32}$$

is a least squares fit of range over a given time interval and

$$\dot{r} = b_1 + 2b_2 t + 3b_3 t^2 + 4b_4 t^3 \tag{33}$$

is a suitable representation for $\dot{r}$, then errors in range and range rate are expressed by

$$\Delta r = \Delta b_0 + t\Delta b_1 + t^2\Delta b_2 + t^3\Delta b_3 + t^4\Delta b_4 \tag{34}$$

$$\Delta\dot{r} = \Delta b_1 + 2t\Delta b_2 + 3t^2\Delta b_3 + 4t^3\Delta b_4 \tag{35}$$

Since time (t) in equations (32) and (33) is not a measured variable, there are no errors in t, and equations (34) and (35) are based on this restriction.

Application of the variance expansion (30) to the sums in the right members of (34) and (35), with time and powers thereof treated as scalars then yields

$$\begin{aligned}\sigma_r^2 = {} & \sigma_{b_0}^2 + t^2\sigma_{b_1}^2 + t^4\sigma_{b_2}^2 + t^6\sigma_{b_3}^2 + t^8\sigma_{b_4}^2 \\ & + 2t\,\mathrm{cov}(b_0, b_1) + 2t^2\,\mathrm{cov}(b_0, b_2) + 2t^3\,\mathrm{cov}(b_0, b_3) \\ & + 2t^4\,\mathrm{cov}(b_0, b_4) + 2t^3\,\mathrm{cov}(b_1, b_2) + 2t^4\,\mathrm{cov}(b_1, b_3) \\ & + 2t^5\,\mathrm{cov}(b_1, b_4) + 2t^5\,\mathrm{cov}(b_2, b_3) + 2t^6\,\mathrm{cov}(b_2, b_4) \\ & + 2t^7\,\mathrm{cov}(b_3, b_4)\end{aligned} \tag{36}$$

and

$$\sigma_{\dot{r}}^2 = \sigma_{b_1}^2 + 4t^2\sigma_{b_2}^2 + 9t^4\sigma_{b_3}^2 + 16t^6\sigma_{b_4}^2$$
$$+ 4t\,\mathrm{cov}(b_1, b_2) + 6t^2\mathrm{cov}(b_1, b_3) + 8t\,\mathrm{cov}(b_1, b_4)$$
$$+ 12t^3\mathrm{cov}(b_2, b_3) + 16t^4\mathrm{cov}(b_2, b_4) + 24t^5\mathrm{cov}(b_3, b_4) \qquad (37)$$

The appropriate covariance and variance terms in the expressions (36) and (37) may be taken from the covariance matrix (29). The resulting formulae for σ_r^2 and $\sigma_{\dot{r}}^2$ are

$$\sigma_r^2 = \sigma_{b_0}^2\Bigg[1 + \frac{2W_0}{W_1}t + \left\{\frac{2W_0}{W_2} + \left(\frac{W_0}{W_1}\right)^2\right\}t^2 + \left\{\frac{2W_0}{W_3} + \frac{2W_0^2}{W_1W_2}\right\}t^3$$
$$+ \left\{\frac{2W_0}{W_4} + \frac{2W_0^2}{W_1W_3} + \left(\frac{W_0}{W_2}\right)^2\right\}t^4 + \left\{\frac{2W_0^2}{W_1W_4} + \frac{2W_0^2}{W_2W_3}\right\}t^5$$
$$+ \left\{\frac{2W_0^2}{W_2W_4} + \left(\frac{W_0}{W_3}\right)^2\right\}t^6 + \frac{2W_0^2}{W_3W_4}t^7 + \left(\frac{W_0}{W_4}\right)^2t^8\Bigg] \qquad (38)$$

$$\sigma_{\dot{r}}^2 = \sigma_b^2\Bigg[\left(\frac{W_0}{W_1}\right)^2 + \frac{4W_0^2}{W_1W_2}t + \left\{4\left(\frac{W_0}{W_2}\right)^2 + \frac{6W_0^2}{W_1W_3}\right\}t^2$$
$$+ \left\{\frac{8W_0^2}{W_1W_4} + \frac{12W_0^2}{W_2W_3}\right\}t^3 + \left\{9\left(\frac{W_0}{W_3}\right)^2 + \frac{16W_0^2}{W_2W_4}\right\}t^4$$
$$+ \frac{24W_0^2}{W_3W_4}t^5 + 16\left(\frac{W_0}{W_4}\right)^2t^6\Bigg] \qquad (39)$$

Corresponding formulae apply for σ_a^2, $\sigma_{\dot{\alpha}}^2$, σ_{ε}^2, and $\sigma_{\dot{\varepsilon}}^2$

2. Equations and Numerical Values for $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, $\sigma_{\dot{\varepsilon}}$

The processes outlined in subheading 1. may now be applied to write out expressions for the standard deviations of range rate errors, azimuth rate errors, and elevation rate errors for a system in which range rate, azimuth rate, and elevation rate are taken as the time derivatives of the least squares representations of range, azimuth, and elevation. The resulting formulae are for:

(a) Third Degree Least Squares Fit

$$\sigma_{\dot{r}}^2 = \sigma_{b_0}^2 \left[\left(\frac{W_0}{W_1}\right)^2 + \frac{4\,W_0^2}{W_1 W_2}\,t + \left\{ 4\left(\frac{W_0}{W_2}\right)^2 + \frac{6\,W_0^2}{W_1 W_3} \right\} t^2 + \frac{12\,W_0^2}{W_2 W_3}\,t^3 + 9\left(\frac{W_0}{W_3}\right)^2 t^4 \right] \tag{40}$$

$$\sigma_{r}^2 = \sigma_{b_0}^2 \left[1 + \frac{2\,W_0}{W_1}\,t + \left\{ \frac{2\,W_0}{W_2} + \left(\frac{W_0}{W_1}\right)^2 \right\} t^2 + \left\{ \frac{2\,W_0}{W_3} + \frac{2\,W_0^2}{W_1 W_2} \right\} t^3 + \left\{ \frac{2\,W_0^2}{W_1 W_3} + \left(\frac{W_0}{W_2}\right)^2 \right\} t^4 + \frac{W_0^2}{W_2 W_3}\,t^5 + \left(\frac{W_0}{W_3}\right)^2 t^6 \right] \tag{41}$$

(b) Fourth Degree Least Squares Fit

$$\sigma_r^2 = \sigma_{b_0}^2 \left[\left(\frac{W_0}{W_1}\right)^2 + \frac{4W_0^2}{W_1 W_2} t + \left\{ 4\left(\frac{W_0}{W_2}\right)^2 + \frac{6W_0^2}{W_1 W_3} \right\} t^2 \right.$$
$$+ \left\{ \frac{8W_0^2}{W_1 W_4} + \frac{12W_0^2}{W_2 W_3} \right\} t^3 + \left\{ 9\left(\frac{W_0}{W_3}\right)^2 + \frac{16W_0^2}{W_2 W_4} \right\} t^4$$
$$\left. + \frac{24W_0^2}{W_3 W_4} t^5 + 16\left(\frac{W_0}{W_4}\right)^2 t^6 \right] \qquad (42)$$

$$\sigma_r^2 = \sigma_{b_0}^2 \left[1 + \frac{2W_0}{W_1} t + \left\{ \frac{2W_0}{W_2} + \left(\frac{W_0}{W_1}\right)^2 \right\} t^2 \right.$$
$$+ \left\{ \frac{2W_0}{W_3} + \frac{2W_0^2}{W_1 W_2} \right\} t^3 + \left\{ \frac{2W_0}{W_4} + \left(\frac{W_0}{W_2}\right)^2 + \frac{2W_0^2}{W_1 W_3} \right\} t^4$$
$$+ \left\{ \frac{2W_0^2}{W_1 W_4} + \frac{2W_0^2}{W_2 W_3} \right\} t^5 + \left\{ \left(\frac{W_0}{W_3}\right)^2 + \frac{2W_0^2}{W_2 W_4} \right\} t^6$$
$$\left. + \frac{2W_0^2}{W_3 W_4} t^7 + \left(\frac{W_0}{W_4}\right)^2 t^8 \right] \qquad (43)$$

(c) Fifth Degree Least Squares Fit

$$\begin{aligned}\sigma_{\dot{r}}^2 = \sigma_{b_0}^2 \Bigg[& \left(\frac{W_0}{W_1}\right)^2 + \frac{4W_0}{W_1 W_2}\, t + \left\{ 4\left(\frac{W_0}{W_2}\right)^2 + \frac{6W_0^2}{W_1 W_3} \right\} t^2 \\ & + \left\{ \frac{8W_0^2}{W_1 W_4} + \frac{12 W_0^2}{W_2 W_3} \right\} t^3 + \left\{ 9\left(\frac{W_0}{W_3}\right)^2 + \frac{16 W_0^2}{W_2 W_4} + \frac{10 W_0^2}{W_1 W_5} \right\} t^4 \\ & + \left\{ \frac{20 W_0^2}{W_2 W_5} + \frac{24 W_0^2}{W_3 W_4} \right\} t^5 + \left\{ 16\left(\frac{W_0}{W_4}\right)^2 + \frac{30 W_0^2}{W_3 W_5} \right\} t^6 \\ & + \frac{40 W_0^2}{W_4 W_5}\, t^7 + 25\left(\frac{W_0}{W_5}\right)^2 t^8 \Bigg]\end{aligned} \tag{44}$$

$$\begin{aligned}\sigma_r^2 = \sigma_{b_0}^2 \Bigg[& 1 + \frac{2W_0}{W}\, t + \left\{ \frac{2W_0}{W} + \left(\frac{W_0}{W}\right)^2 \right\} t^2 \\ & + \left\{ \frac{2W_0}{W_3} + \frac{W_0^2}{W_1 W_2} \right\} t^3 + \left\{ \frac{2W_0}{W_4} + \frac{2W_0^2}{W_1 W_3} + \left(\frac{W_0}{W_2}\right)^2 \right\} t^4 \\ & + \left\{ \frac{2W_0}{W_5} + \frac{2W_0^2}{W_1 W_4} + \frac{2W_0^2}{W_2 W_3} \right\} t^5 + \left\{ \left(\frac{W_0}{W_3}\right)^2 + \frac{2W_0^2}{W_1 W_5} + \frac{2W_0^2}{W_2 W_4} \right\} t^6 \\ & + \left\{ \frac{2W_0^2}{W_2 W_5} + \frac{2W_0^2}{W_3 W_4} \right\} t^7 + \left\{ \left(\frac{W_0}{W_4}\right)^2 + \frac{2W_0^2}{W_3 W_5} \right\} t^8 \\ & + \frac{2W_0^2}{W_4 W_5}\, t^9 + \left(\frac{W_0}{W_5}\right)^2 t^{10} \Bigg]\end{aligned} \tag{45}$$

Almost identical formulae apply for $\sigma_{\dot{\alpha}}^2$, σ_{α}^2, $\sigma_{\dot{\varepsilon}}^2$, σ_{ε}^2 with $\sigma_{b_o}^2$ replaced by $\sigma_{a_o}^2$ and $\sigma_{c_o}^2$.

Each pair of equations in the group (40),.........,(45) may be abbreviated to read

$$\sigma_r = \sigma_{b_o}\sqrt{F_1(t)} \tag{46}$$

$$\sigma_{\dot{r}} = \sigma_{b_o}\sqrt{F_2(t)} \tag{47}$$

where $F_1(t)$ and $F_2(t)$ are polynomials in the time. For a given interval ($1 \leq t \leq 10$) of least squares fitting σ_{b_o} is a constant and we may take

$$\sigma_{b_o} = \frac{(\sigma_r)_{Mean}}{\left(\sqrt{F_1(t)}\right)_{Mean}} \tag{48}$$

where $\sigma_{r_{Mean}}$ is the standard deviation calculated numerically in Part I. With σ_{b_o} known, $\sigma_{\dot{r}}$ vs. time over the interval ($1 \leq t \leq 10$) may be computed with (47). Formulae similar to (48) apply for σ_{a_o} and σ_{c_o} . In each case $\left(\sqrt{F_1(t)}\right)_{Mean}$ is calculated arithmetically by taking values of $\sqrt{F_1(t)}$ at several points in the interval ($1 \leq t \leq 10$).

When this procedure is applied to the Woomera Station Data of MA-6, the variation of $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, and $\sigma_{\dot{\varepsilon}}$ with time is as shown in Figures 5, 6, 7, 8, 9. The averages of $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, and $\sigma_{\dot{\epsilon}}$ (with respect to time) are shown in Figures 10, 11, and 12.

The extreme dependence of calculated $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, and $\sigma_{\dot{\epsilon}}$ on time illustrated in Figures 5, 6, 7, 8, 9 resulted from the fact that a small sample size (10 per minute) was used to determine the least squares fittings of range, azimuth, and elevation. Values of $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, and $\sigma_{\dot{\epsilon}}$ should be constant, if these quantities are to be useful for engineering purposes. In part IV of this report it is shown that the difficulties presented in these figures may be overcome by using a larger sample size to determine the least squares fittings of range, azimuth, and elevation.

IV. Dependence of $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, $\sigma_{\dot{\varepsilon}}$, σ_r , σ_α , σ_ε on the Number of Samples Used for the Least Squares Fit

The methods of Part III, when applied to the calculation of $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, $\sigma_{\dot{\varepsilon}}$ for range rate, azimuth rate, and elevation rate, showed that these standard deviations vary rather radically over an interval for which r , α , ε , have been fitted by least squares polynomials. Since all of the calculations for Parts II and III are based on least squares fitting with ten available samples per minute, it is natural to inquire whether the values of $\sigma_{\dot{r}}$, $\sigma_{\dot{\alpha}}$, $\sigma_{\dot{\varepsilon}}$ as well as σ_r , σ_α , σ_ε could be improved by using a larger number of samples. To answer this question, it will suffice to examine only the fourth degree least squares fittings. The results of this part are not dependent on Woomera station data and will apply to any tracking station.

The formulae for σ_r^2 and $\sigma_{\dot{r}}^2$ for a fourth degree least squares fit of range are (Part III, equations (42) and (43)):

$$\sigma_r^2 = \sigma_{b_0}^2\Bigg[1 + \frac{2W_0}{W} + \left\{\frac{2W_0}{W_2} + \left(\frac{W_0}{W_1}\right)^2\right\}t^2 + \left\{\frac{2W_0}{W_3} + \frac{2W_0^2}{W_1W_2}\right\}t^3$$
$$+ \left\{\frac{2W_0}{W_4} + \left(\frac{W_0}{W_2}\right)^2 + \frac{2W_0^2}{W_1W_3}\right\}t^4 + \left\{\frac{2W_0^2}{W_1W_4} + \frac{2W_0^2}{W_2W_3}\right\}t^5$$
$$+ \left\{\left(\frac{W_0}{W_3}\right)^2 + \frac{2W_0^2}{W_2W_4}\right\}t^6 + \frac{2W_0^2}{W_3W_4}t^7 + \left(\frac{W_0}{W_4}\right)^2 t^8\Bigg] \tag{49}$$

$$\sigma_{\dot{r}}^2 = \sigma_{b_0}^2 \Bigg[\left(\frac{W_0}{W_1}\right)^2 + \frac{4W_0^2}{W_1 W_2} t + \left\{ 4\left(\frac{W_0}{W_2}\right)^2 + \frac{6W_0^2}{W_1 W_3} \right\} t^2 + \left\{ \frac{8W_0}{W_1 W_4} + \frac{12 W_0^2}{W_2 W_3} \right\} t^3 + \left\{ 9\left(\frac{W_0}{W_3}\right)^2 + \frac{16 W_0^2}{W_2 W_4} \right\} t^4 + \frac{24 W_0^2}{W_3 W_4} t^5 + 16\left(\frac{W_0}{W_4}\right)^2 t^6 \Bigg] \tag{50}$$

Recall that these formulae hold over a one minute interval over which range has been fitted by a fourth degree least squares polynomial.

The coefficients $\frac{W_0}{W_1}$, $\frac{W_0}{W_2}$,.........., etc. in equations (49) and (50) may be written as approximations in terms of the number of samples. Consider the fittings for range when the total variation of r* over a one minute interval is not large. N is the number of values of r* used to make the least squares fit.

Then[3]

$$\frac{W_0}{W_1} = \frac{\sum_{j=1}^{N} r_j^*}{\sum_{j=1}^{N} j r_j^*} \doteq \frac{N}{\sum_{j=1}^{N} j} = \frac{N}{\frac{N(N+1)}{2}} = \frac{2}{N+1}$$

$$\frac{W_0}{W_2} = \frac{\sum_{j=1}^{N} r_j^*}{\sum_{j=1}^{N} j^2 r_j^*} \doteq \frac{N}{\sum_{j=1}^{N} j^2} = \frac{N}{\frac{N(N+1)(2N+1)}{6}} = \frac{6}{(N+1)(2N+1)}$$

$$\frac{W_0}{W_3} = \frac{\sum_{j=1}^{N} r_j^*}{\sum_{j=1}^{N} j^3 r_j^*} \doteq \frac{N}{\sum_{j=1}^{N} j^3} = \frac{N}{\left\{\frac{N(N+1)}{2}\right\}^2} = \frac{4}{N(N+1)^2}$$

$$\frac{W_0}{W_4} = \frac{\sum_{j=1}^{N} r_j^*}{\sum_{j=1}^{N} j^4 r_j^*} \doteq \frac{N}{\sum_{j=1}^{N} j^4} = \frac{N}{\left\{\frac{N(N+1)(2N+1)}{6}\right\}\left\{\frac{3N(N+1)-1}{5}\right\}}$$

$$= \frac{30}{(N+1)(2N+1)\left\{3N(N+1)-1\right\}}$$

[3] From reference 10, p. 387:

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2} \; ; \quad \sum_{j=1}^{N} j^2 = \frac{N(N+1)(N+2)}{6} \; ;$$

$$\sum_{j=1}^{N} j^3 = \left\{\frac{N(N+1)}{2}\right\}^2 \; ; \quad \sum_{j=1}^{N} j^4 = \left\{\frac{N(N+1)(N+2)}{6}\right\}\left\{\frac{3N(N+1)-1}{5}\right\}$$

$$\frac{W_0^2}{W_1W_2} \doteq \frac{12}{(N+1)^2(2N+1)}$$

$$\frac{W_0^2}{W_2W_3} \doteq \left\{\frac{6}{(N+1)(2N+1)}\right\}\left\{\frac{4}{N(N+1)^2}\right\} = \frac{24}{N(N+1)^3(2N+1)}$$

Etc...............Etc.

When all of the coefficients in equation (49) and (50) are approximated in this manner in terms of the number of samples, N, the resulting approximations for σ_r^2 and $\sigma_{\dot{r}}^2$ are

$$\begin{aligned}\sigma_r^2 = \sigma_{b_0}^2 \Bigg[1 &+ \frac{4}{N+1}t + \left\{\frac{12}{(N+1)(2N+1)} + \frac{4}{(N+1)^2}\right\}t^2 \\ &+ \left\{\frac{8}{N(N+1)^2} + \frac{24}{(N+1)^2(2N+1)}\right\}t^3 \\ &+ \left\{\frac{20}{N(N+1)^2(2N+1)} + \frac{36}{(N+1)^2(2N+1)^2} + \frac{16}{N(N+1)^3}\right\}t^4 \\ &+ \left\{\frac{88}{N(N+1)^3(2N+1)}\right\}t^5 + \left\{\frac{16}{N^2(N+1)^4} + \frac{120}{N(N+1)^3(2N+1)^2}\right\}t^6 \\ &+ \left\{\frac{80}{N^2(N+1)^4(2N+1)}\right\}t^7 + \left\{\frac{100}{N^2(N+1)^4(2N+1)^2}\right\}t^8 \Bigg]\end{aligned} \qquad (51)$$

$$\sigma_{\dot{r}}^{2} = \sigma_{b_0}^{2}\left[\frac{4}{(N+1)^2} + \left\{\frac{48}{(N+1)^2(2N+1)}\right\} t + \left\{\frac{144}{(N+1)^2(2N+1)^2} + \frac{45}{N(N+1)^3}\right\} t^2 + \left\{\frac{448}{N\,(2N+1)(N+1)^3}\right\} t^3 + \left\{\frac{144}{N^2(N+1)^4} + \frac{640}{N\,(2N+1)^2(N+1)^3}\right\} t^4 + \left\{\frac{960}{N^2(2N+1)(N+1)^4}\right\} t^5 + \left\{\frac{1600}{N^2(2N+1)^2(N+1)^4}\right\} t^6 \right] \quad (52)$$

In these formulae the further simplification $3N(N+1)-1 \approx 3N(N+1)$ has been made. The formulae apply over an interval $(1 \leq t \leq 10)$ for which the range r has been fitted by a fourth degree least squares polynomial.

The approximation formula (51) applies to σ_{α}^{2} when σ_{b_0} is replaced by σ_{a_0}, and to σ_{ε}^{2} when σ_{b_0} is replaced by σ_{c_0}. Similarly, the approximation (52) applies to $\sigma_{\dot{\alpha}}^{2}$ when σ_{b_0} is replaced by σ_{a_0}, and to $\sigma_{\dot{\varepsilon}}^{2}$ when σ_{b_0} is replaced by σ_{c_0}

Figure 13 illustrates the dependence of the average values of normalized standard deviations $\frac{\sigma_r}{\sigma_{b_0}}$ $\frac{\sigma_{\alpha}}{\sigma_{a_0}}$, and $\frac{\sigma_{\varepsilon}}{\sigma_{c_0}}$

on the number N of samples used to make the least squares fit of r, α, or ε. As formula (51) implies, for large N $\frac{\sigma_{\dot{r}}}{\sigma_{b_0}}$, $\frac{\sigma_{\dot{\alpha}}}{\sigma_{a_0}}$, and $\frac{\sigma_{\dot{\varepsilon}}}{\sigma_{c_0}}$ become independent of time and $\lim_{N\to\infty}(\sigma_{\dot{r}}) = \sigma_{b_0}$, $\lim_{N\to\infty}(\sigma_{\dot{\alpha}}) = \sigma_{a_0}$, and $\lim_{N\to\infty}(\sigma_{\dot{\varepsilon}}) = \sigma_{c_0}$.

Figure 14 illustrates the dependence of the average values of normalized standard deviations $\frac{\sigma_{\dot{r}}}{\sigma_{b_0}}$, $\frac{\sigma_{\dot{\alpha}}}{\sigma_{a_0}}$, and $\frac{\sigma_{\dot{\varepsilon}}}{\sigma_{c_0}}$ on the number N of samples used to make the least squares fit of range, azimuth, or elevation over a one minute interval. It is readily observed that the normalized deviations of range rate errors, azimuth rate errors, and elevation rate errors may be dramatically reduced by increasing the sample size used to make the least squares fit of r, α, or ε. In fact, from formula (52),

$$\lim_{N\to\infty}\left(\frac{\sigma_{\dot{r}}}{\sigma_{b_0}}\right) = \lim_{N\to\infty}\left(\frac{\sigma_{\dot{\alpha}}}{\sigma_{a_0}}\right) = \lim_{N\to\infty}\left(\frac{\sigma_{\dot{\varepsilon}}}{\sigma_{c_0}}\right) = 0$$

The formulae (51) and (52) used the assumption that the range of variation of range, azimuth, and elevation over a one minute interval is not large. In case this assumption is not feasible, the same procedure used here may be employed, but with an additional parameter $\varsigma = \frac{r_{Max}}{r_{Min}}$ in the expressions for the ratios $\frac{W_2}{W_1}$, $\frac{W_0}{W_2}$, etc. The relations developed in this part are intended to illustrate approximate dependence of the error standard deviations on sample size N.

When the results of this part are compared with the material in Part III it is clear that for N sufficiently large (and only then) the off diagonal elements in the covariance matrix, equation (29), approach zero; in addition all diagonal elements except that in the upper left hand corner become vanishingly small. Under these circumstances, the error standard deviations σ_r, σ_α, and σ_ε approach constant values σ_{b_0} , σ_{a_0} , and σ_{c_0} , respectively for the interval of least squares fit.

Howard H. Brown

Howard H. Brown
Staff Member
Plans Office

F O Vonbun

F. O. Vonbun
Head, Plans Office

Acknowledgments

The numerican data processing of Part II to obtain least squares fits and standard deviations of range, azimuth, and elevation were carried out on the GSFC 7090 computer by Mr. J. J. Streeter of the Advanced Orbital Programming Group. The computations for Parts III and IV were performed by Mrs. Aileen Marlow of the Plans Office.

References

1. NASA Project Mercury Computing Memorandum No. 3, "Summary of MA-6 Computing and Data Flow Results," April 6, 1962.

2. Alexander, Howard W., Elements of Mathematical Statistics, Wiley & Sons, Inc., 1961.

3. Parzen, Emanuel, Modern Probability Theory and Its Applications, Wiley & Sons, Inc., 1960.

4. Grossmann, Walter, Grundzüge der Ausgleichungsrechnung, Springer-Verlag, 1961.

5. Povejsil, D. J., Raven, R. S., Waterman, P., Airborne Radar - Principles of Guided Missile Design, D. Van Nostrand Company, Inc., 1961.

6. Feller, W., Probability Theory and Its Applications, Volume I, John Wiley & Sons, Inc., 1950.

7. Worthing, A. G., Geffner, J., Treatment of Experimental Data, John Wiley & Sons, Inc., 1959.

8. Milne, W. E., Numerical Calculus, Princeton University Press, 1949.

9. Scarborough, J. B., Numerical Mathematical Analysis, Fourth Edition, The Johns Hopkins Press, 1958.

10. Hodgman, C. D., Selby, S. M., Weast, R. C., C. R. C. Standard Mathematical Tables, Twelfth Edition, Chemical Rubber Publishing Company, 1959.

11. Arley, N., Buch, K. R., Introduction to the Theory of Probability and Statistics, John Wiley & Sons, Inc., 1950.

12. Burington, R. S., May, D. C., Handbook of Probability and Statistics with Tables, Handbook Publishers, Inc., 1958.

Appendix A

Linear Relations Between Least Squares Coefficients

The computation of covariance elements in the covariance matrix of a set of least squares coefficients is considerably simplified by taking advantage of simple linear relations which connect any pair of least squares coefficients. These relations will now be derived for the case of a fourth degree least squares polynomial. The results extend by induction to a least squares polynomial of any degree.

For the case of a fourth degree least squares polynomial fit for range we have

$$\dot{r} = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 \tag{A-1}$$

The least squares coefficients b_0, b_1, b_2, b_3, b_4 are determined by the relation

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{12} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{13} & k_{23} & k_{33} & k_{34} & k_{35} \\ k_{14} & k_{24} & k_{34} & k_{44} & k_{45} \\ k_{15} & k_{25} & k_{35} & k_{45} & k_{55} \end{bmatrix} \begin{bmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} \tag{A-2}$$

where:

$$w_0 = \sum_{j=1}^{n} r_j^* \qquad w_1 = \sum_{j=1}^{n} j\, r_j^* \qquad w_2 = \sum_{j=1}^{n} j^2 r_j^*$$

$$w_3 = \sum_{j=1}^{n} j^3 r_j^* \qquad w_4 = \sum_{j=1}^{n} j^4 r_j^*$$

The quantities $r_j^* (j = 1, 2, \ldots, n)$ are the measured values of range recorded at equi-spaced times. The 5x5 matrix (K_{ij}) is the inverse of the 5x5 matrix which appears on the left side of equation (5). The forms of the indices on individual elements of the matrix (k_{ij}) in equation (A-2) follow from the fact that the matrix (K_{ij}) is a symmetric matrix.

When the matrix equation (A-2) is written out for the individual coefficients the following linear equations are obtained

$$b_0 = K_{11}w_0 + K_{12}w_1 + K_{13}w_2 + K_{14}w_3 + K_{15}w_4 \tag{A-3}$$

$$b_1 = K_{12}w_0 + K_{22}w_1 + K_{23}w_2 + K_{24}w_3 + K_{25}w_4 \tag{A-4}$$

$$b_2 = K_{13}w_0 + K_{23}w_1 + K_{33}w_2 + K_{34}w_3 + K_{35}w_4 \tag{A-5}$$

$$b_3 = K_{14}w_0 + K_{24}w_1 + K_{34}w_2 + K_{44}w_3 + K_{45}w_4 \tag{A-6}$$

$$b_4 = K_{15}w_0 + K_{25}w_1 + K_{35}w_2 + K_{45}w_3 + K_{55}w_4 \tag{A-7}$$

Elimination of K_{12} between equations (A-3) and (A-4) yields

$$b_1 = \left(\frac{W_0}{W_1}\right) b_0 + \left(k_{22}W_1 + k_{23}W_2 + k_{24}W_3 + k_{23}W_4 - \frac{W_0^2}{W_1} K_{11} - \frac{W_0 W_2 K_{13}}{W_1} - \frac{W_0 W_3 K_{14}}{W_1} - \frac{W_0 W_4 k_{15}}{W_1} \right)$$

The last parenthesis term in this equation is a constant. The relation between b_1 and b_0 is therefore of the form

$$b_1 = \left(\frac{W_0}{W_1}\right) b_0 + C_1 \tag{A-8}$$

Similarly, elimination of K_{13} between equations (A-3) and (A-5) yields the linear relation

$$b_2 = \left(\frac{W_0}{W_2}\right) b_0 + C_2 \tag{A-9}$$

Elimination of k_{23} between equations (A-4) and (A-5) yields

$$\begin{aligned} b_2 &= \left(\frac{W_1}{W_2}\right) b_1 + \left(k_{13}W_0 + k_{33}W_2 + k_{34}W_3 + k_{35}W_4 - \frac{W_0 W_1 K_{12}}{W_2} - \frac{W_1^2 k_{22}}{W_2} - \frac{W_1 W_3 k_{24}}{W_2} - \frac{W_1 W_4 k_{25}}{W_2} \right) \\ &= \left(\frac{W_1}{W}\right) b_1 + C_{21} \end{aligned} \tag{A-10}$$

where C_{21} is a constant.

Proceeding by induction from equations (A-8) and (A-9), it may be shown that any least squares coefficient b_p in a set of least squares coefficients ($b_0, b_1, b_2, \ldots$) may be expressed linearly in terms of b_0 in the form

$$b_P = \left(\frac{W_0}{W_P}\right) b_0 + C_P \tag{A-11}$$

For a fourth degree least squares fit with least squares coefficients ($b_0, b_1, \ldots, b_4$), index P in (A-11) has the values 1, 2, 3, or 4.

Further, equation (A-10) may be generalized so that any least squares coefficient b_P is related to any other lease squares coefficient b_ℓ ($\ell \neq P$) of the same set by a linear equation of form

$$b_P = \left(\frac{W_\ell}{W_P}\right) b_\ell + C_{P\ell} \tag{A-12}$$

where $C_{P\ell}$ is a constant for each pair (P, ℓ).

The linear relations developed here all follow from the fact that the matrix (K_{ij}) in (A-2) is always a symmetric matrix.

Table I

STANDARD DEVIATION - 27949006 00

THE FOLLOWING DATA HAS BEEN FITTED TO A POLY. OF DEGREE 04

THE INPUT PARAMETERS ARE. K- 004 N=- 010 N- 010 F- 000000000

96109	4TEXVL	600901	1718	060000	3491504	128211	094583380	000000	000	1
96109	4TEXVL	600901	1718	120000	3513994	135037	091533193	000000	000	1
96109	4TEXVL	600901	1718	180000	3538111	141730	088628380	000000	000	1
96109	4TEXVL	600901	1718	240000	3563996	148340	085884941	000000	000	1
96109	4TEXVL	6C0901	1718	300000	3591743	154866	083319314	000000	000	1
96109	4TEXVL	600901	1718	360000	3621402	161259	080948375	000000	000	1
96109	4TEXVL	600901	1718	420000	3652977	167418	078789438	000000	000	1
96109	4TEXVL	600901	1718	480000	3686429	173194	076860260	000000	000	1
96109	4TEXVL	600901	1718	540000	3721672	178388	075179032	000000	000	1
96109	4TEXVL	600901	1719	000000	3758576	182753	073764380	000000	000	1

THE FOLLOWING DATA HAS BEEN PROCESSED POINT FOR POINT

96109	4TEXVL	600901	1719	060000	0196980	186440	072629350	000000	000	1
96109	4TEXVL	600901	1719	120000	0236670	189130	071788790	000000	000	1
96109	4TEXVL	600901	1719	180000	0277210	190810	071253460	000000	000	1
96109	4TEXVL	600901	1719	240000	0318270	191440	071029040	000000	000	1
96109	4TEXVL	600901	1719	300000	0359310	190780	071119650	000000	000	1
96109	4TEXVL	6C0901	1719	360000	0399980	188940	071523480	000000	000	1
96109	4TEXVL	600901	1719	420000	0440060	186050	072235920	000000	000	1
96109	4TEXVL	600901	1719	480000	0479000	182200	073247590	000000	000	1
96109	4TEXVL	600901	1719	540000	0516470	177680	074546070	000000	000	1
96109	4TEXVL	600901	1720	000000	0552310	172290	076116970	000000	000	1

AZIMUTH

O	C	O-C	SUM(O-C)**
19697999 02	19697999 02	71525573-06	51159075-12
23666999 02	23666622 02	37717819-03	14226389-06
27720999 02	27722790 02	-17910003-02	33499461-05
31826999 02	31824284 02	27146339-02	10719183-04
35930999 02	35930549 02	45013427-03	10921804-04
39997999 02	40002685 02	-46858787-02	32879263-04
44005999 02	44003452 02	25463104-02	39362960-04
47899999 02	47897275 02	27232170-02	46778871-04
51646999 02	51650234 02	-32353401-02	57246296-04
55230999 02	55230072 02	92744827-03	58106456-04

STANDARD DEVIATICN - 34090015-02

σ_α(MILS)=17.45 X .0034

ELEVATION

O	C	O-C	SUM(O-C)**
18643999 02	18642040 02	19593238-02	38389500-05
18912999 02	18916380 02	-33810138-02	15270204-04
19080999 02	19084280 02	-32808780-02	26034365-04
19143999 02	19138041 02	59583187-02	61535926-04
19077999 02	19074725 02	32744407-02	72257888-04
18893999 02	18896155 02	-21562576-02	76907335-04
18604999 02	18608919 02	-39198398-02	92272479-04
18219999 02	18224363 02	-43635368-02	11131293-03
17767999 02	17758594 02	94046592-02	19976054-03
17228999 02	17232485 02	-34861564-02	21191383-03

STANDARD DEVIATICN - 65102048-02

σ_ϵ(MILS)=17.45 X .0065

RANGE

O	C	O-C	SUM(O-C)**
72629348 05	72629403 05	-54687500-01	29907226-02
71788788 05	71788744 05	43945312-01	49219131-02
71253458 05	71253216 05	24218750 00	63576698-01
71029038 05	71029320 05	-28222656 00	14322853 00
71119648 05	71119764 05	-11621093 00	15673351 00
71523478 05	71523474 05	39062500-02	15674877 00
72235918 05	72235579 05	33984375 00	27224254 00
73247588 05	73247426 05	16210937 00	29852199 00
74546068 05	74546567 05	-49902343 00	54754638 00
76116968 05	76116768 05	20019531 00	58762454 00

STANDARD DEVIATICN - 34281906 00

σ_r(METER)=.9144 X .34

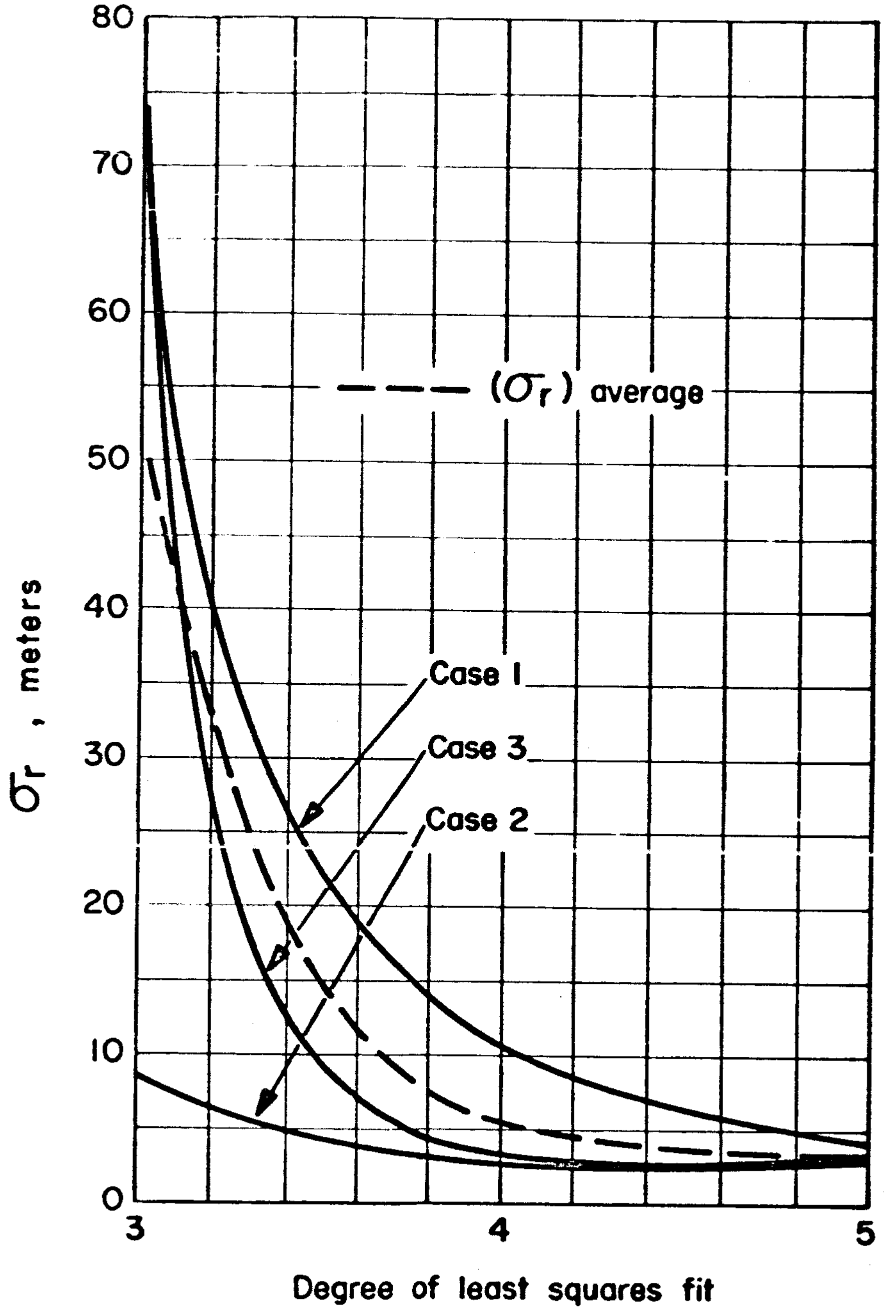


Figure 1 Standard deviation of range errors vs degree of least squares fit of range.

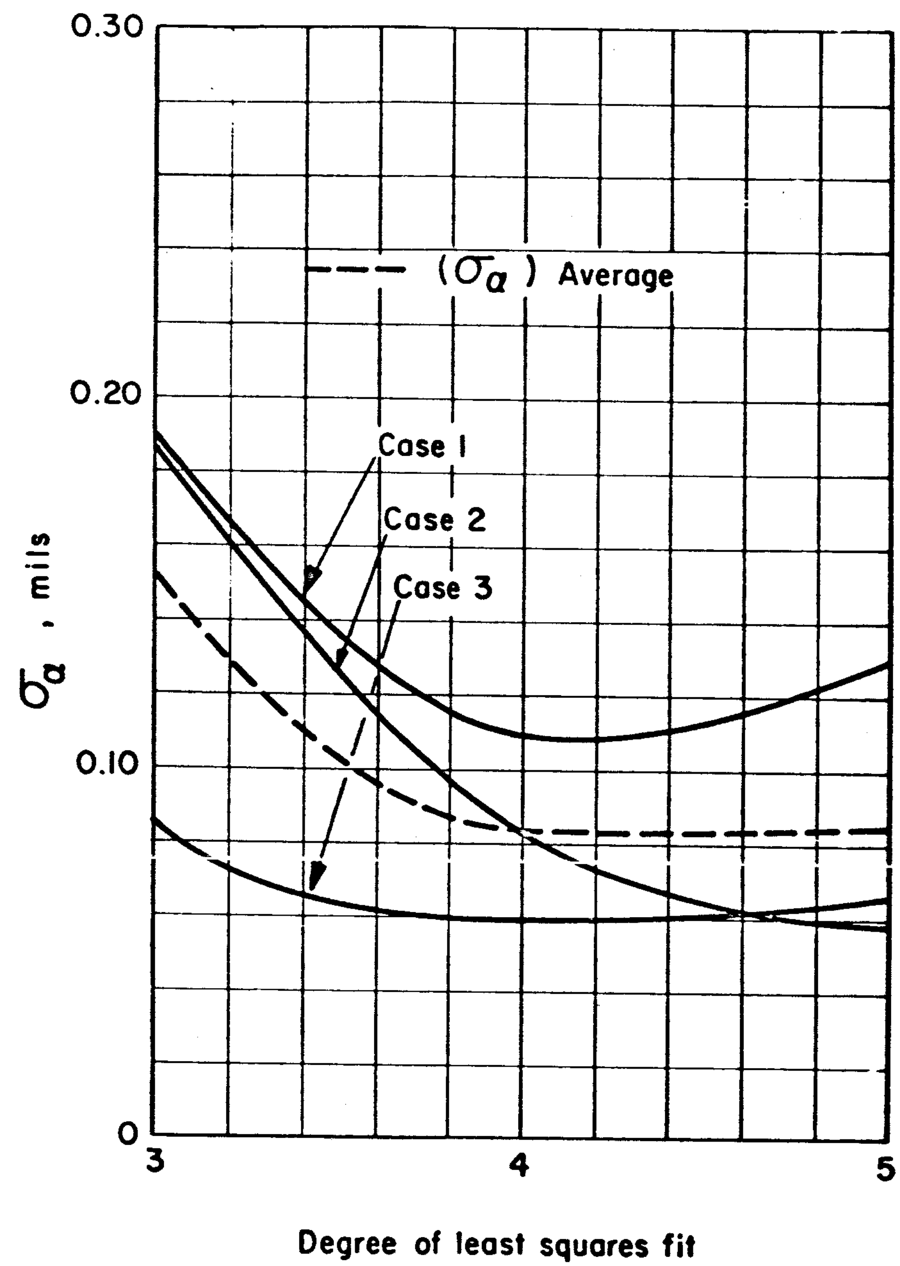


Figure 2 Standard deviation of azimuth errors vs degree of least squares fit of azimuth.

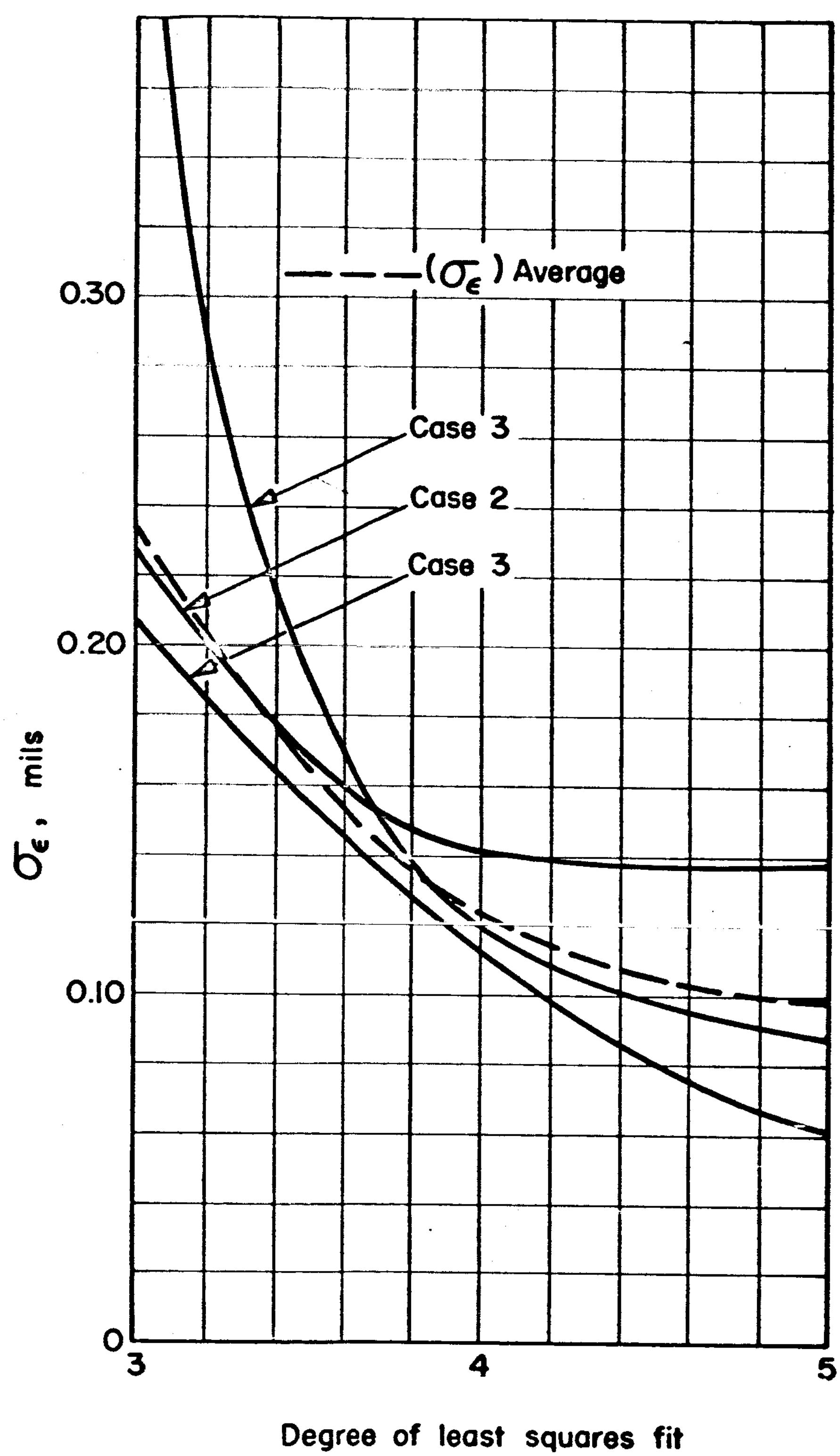
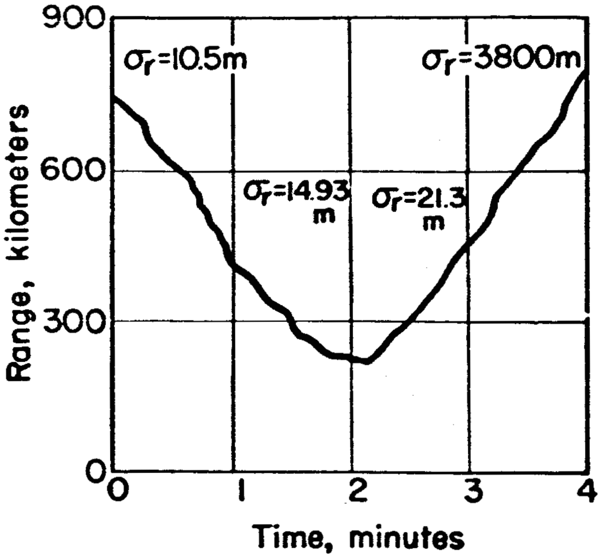


Figure 3 Standard deviation of elevation errors vs degree of least squares fit of elevation.

WOOMERA STATION, MA-6 MISSION



Note - For explanation of large values of σ_r, σ_a, σ_ϵ see discussion of dynamic system errors, pages 13, 14.

Values of σ_r, σ_a, σ_ϵ apply over an interval 0.9 minute long, measured to the left from 1 min., 2 min., 3 min., 4 min. respectively.

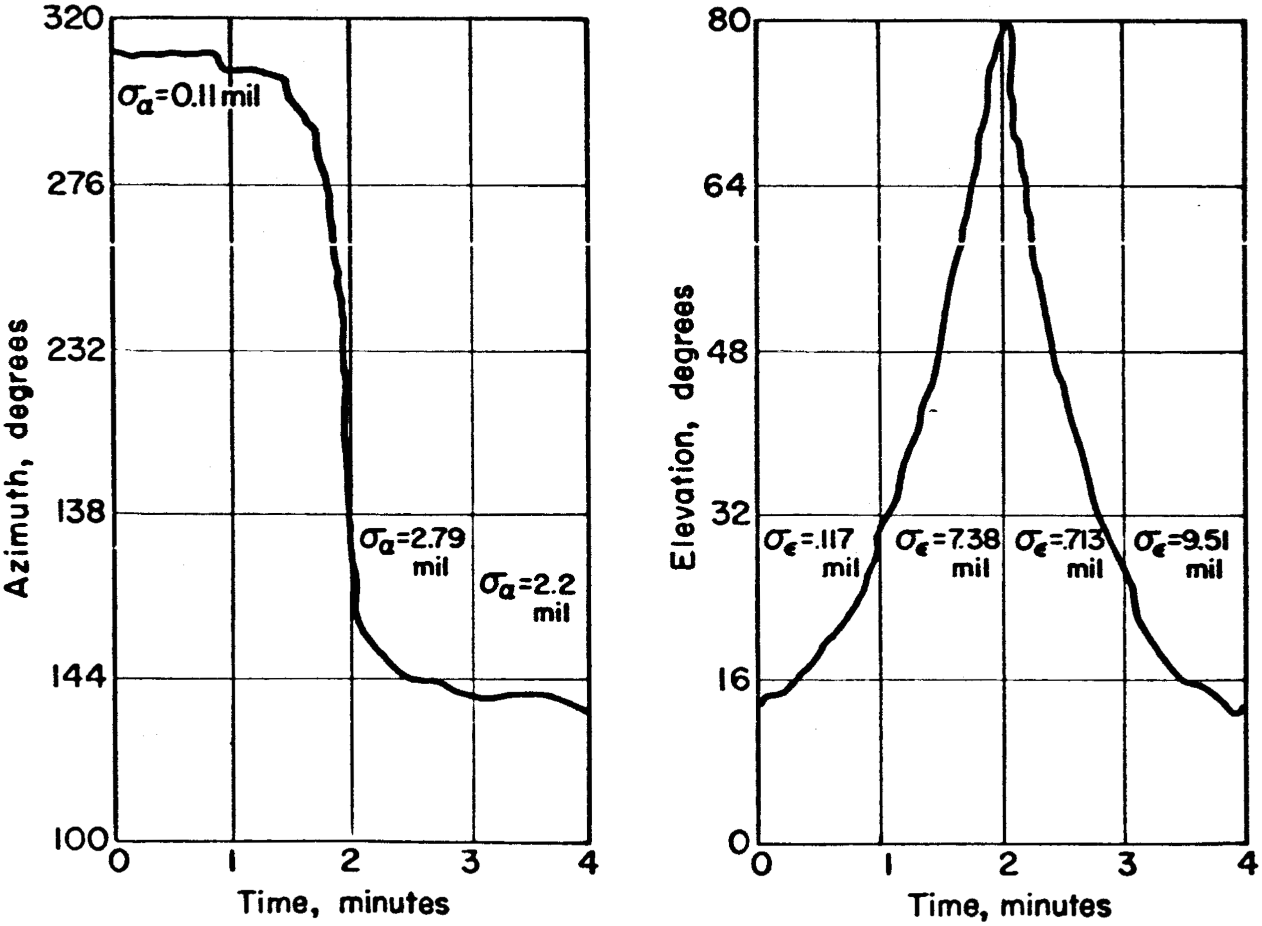
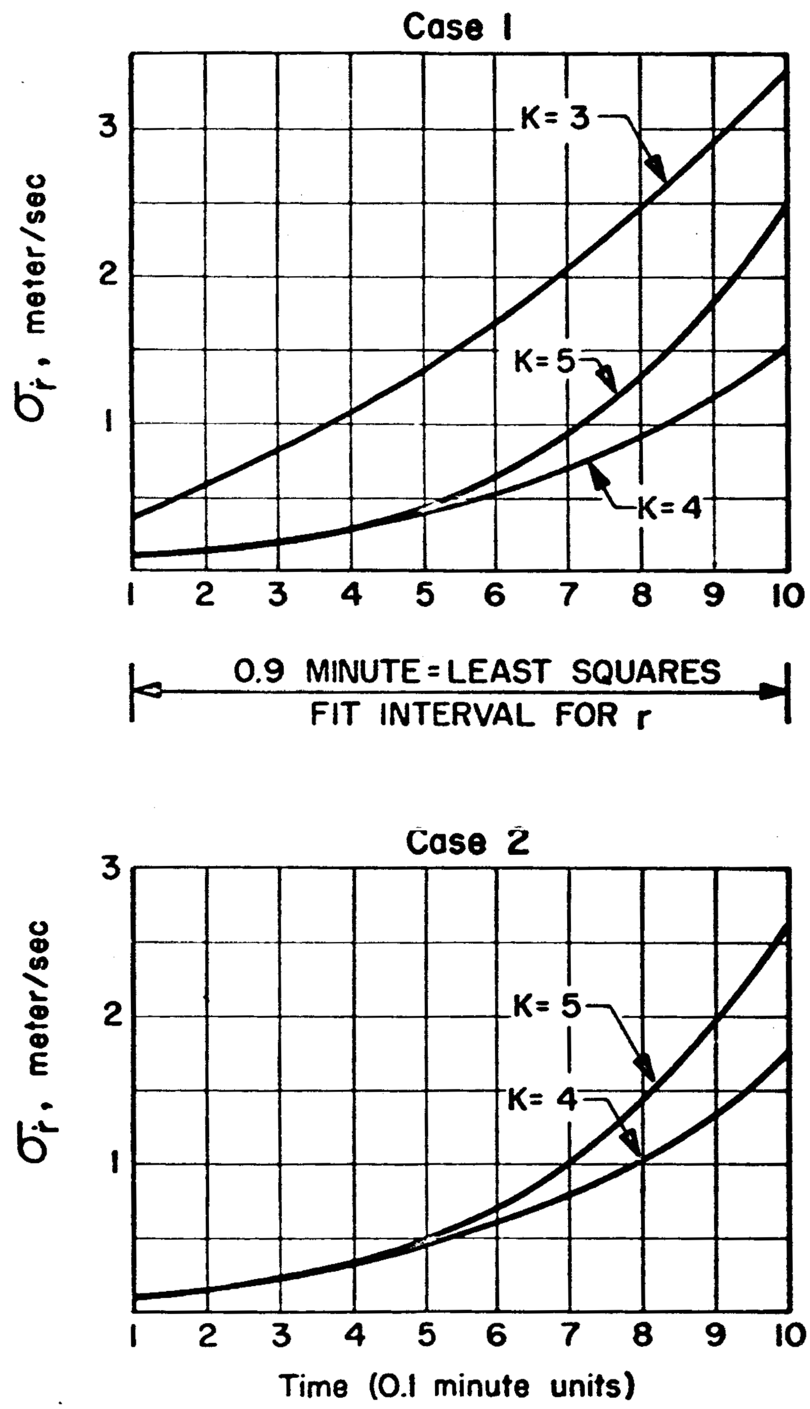


Figure 4 Dependence of standard deviations of range, azimuth, and elevation errors on shape of range, azimuth, and elevation vs time curves. σ's shown for 4th degree least square fit.

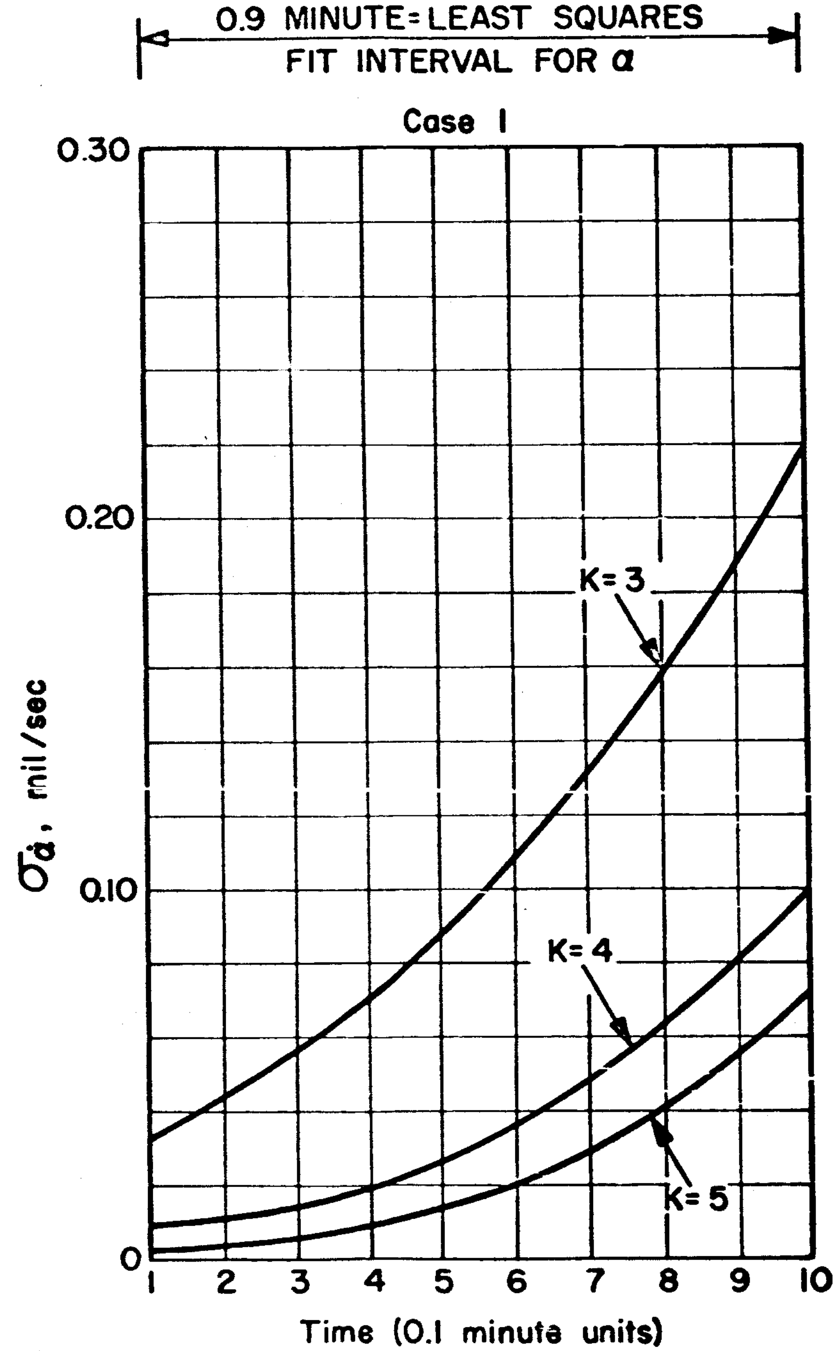
Values of $\sigma_{\dot{r}}$ should be constant. The variation of $\sigma_{\dot{r}}$ with time indicated here results from the small sample size (10 per minute) used. For improvement in $\sigma_{\dot{r}}$ via use of larger sample size see discussion in part IV.



K= Degree of least squares fit of r

Figure 5 Variation of standard deviation of range rate errors over an interval of least squares fit of range. Values of $\dot{r}$ determined from time derivative of least squares fit of range.

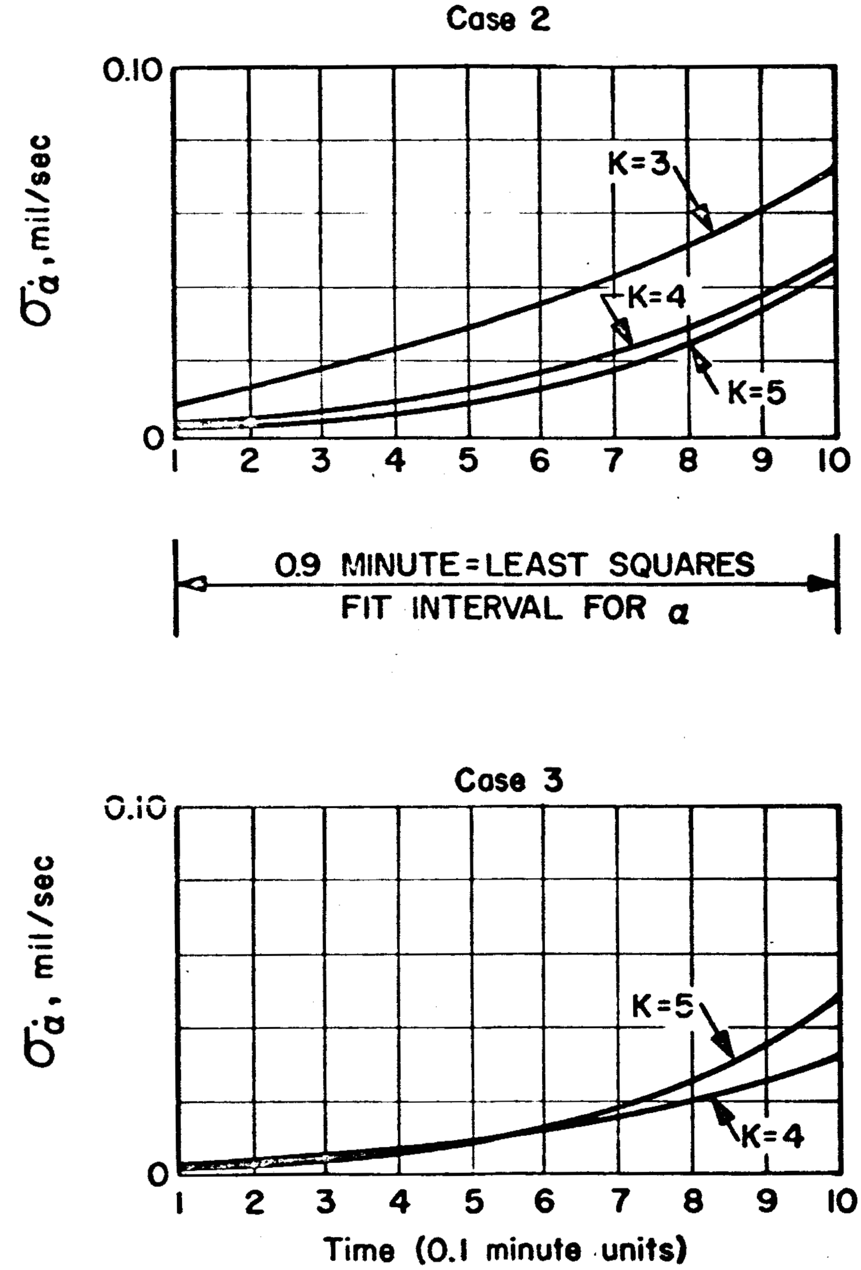
Values of $\sigma_{\dot{\alpha}}$ should be constant. The variation of $\sigma_{\dot{\alpha}}$ with time indicated here results from the small sample size (10 per minute) used. For improvement in $\sigma_{\dot{\alpha}}$ via use of larger sample size see discussion in part IV.



K= Degree of least squares fit for α

Figure 6 Variation of standard deviation of azimuth rate errors over an interval of least squares fit of azimuth. Values of $\dot{\alpha}$ determined from time derivative of least square fit of azimuth.

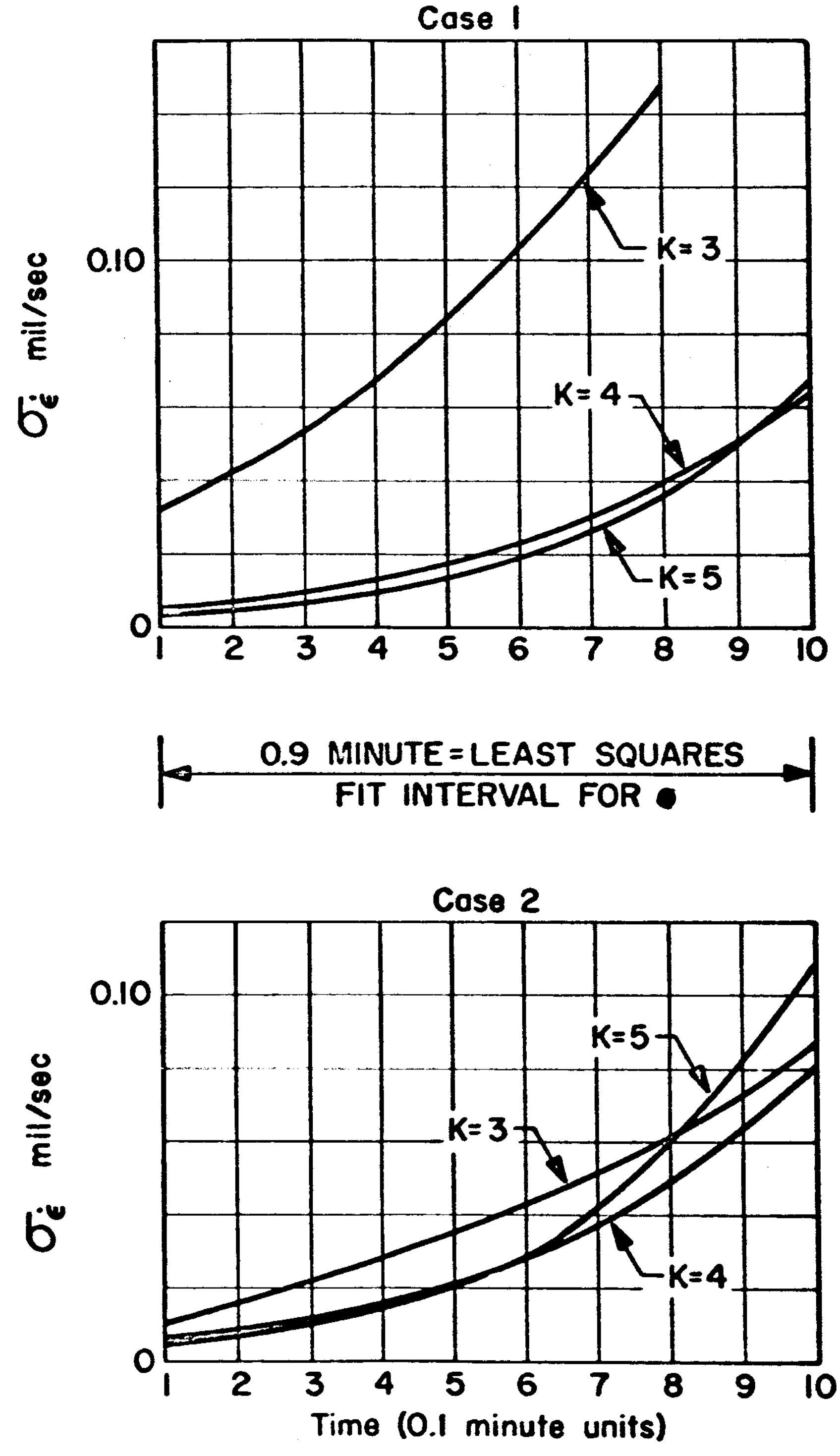
Values of $\sigma_{\dot{\alpha}}$ should be constant. The variation of $\sigma_{\dot{\alpha}}$ with time indicated here results from the small sample size (10 per minute) used. For improvement in $\sigma_{\dot{\alpha}}$ via use of larger sample size see discussion in part IV.



K = Degree of least squares fit for α

Figure 7 Variation of standard deviation of azimuth rate errors over an interval of least squares fit of azimuth. Values of $\dot{\alpha}$ determined from time derivative of least square fit of azimuth.

Values of $\sigma_{\dot{\epsilon}}$ should be constant. The variation of $\sigma_{\dot{\epsilon}}$ with time indicated here results from the small sample size (10 per minute) used. For improvement in $\sigma_{\dot{\epsilon}}$ via use of larger sample size see discussion in part IV.

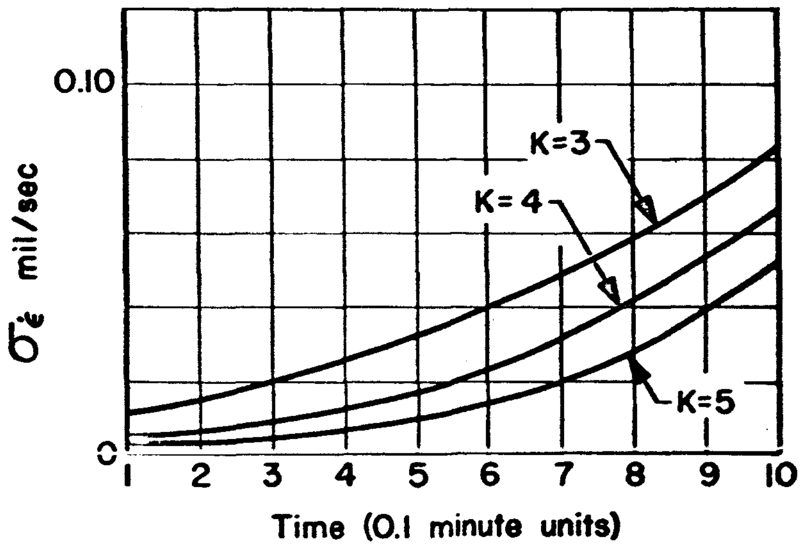


K= Degree of least squares fit of ●

Figure 8 Variation of standard deviation of elevation rate errors over an interval of least squares fit of range. Values of $\dot{\epsilon}$ determined from least squares fit of elevation.

Values of $\sigma_{\dot{\epsilon}}$ should be constant. The variation of $\sigma_{\dot{\epsilon}}$ with time indicated here results from the small sample size (10 per minute) used. For improvement in $\sigma_{\dot{\epsilon}}$ via use of larger sample size see discussion in part IV.

Case 3



K= Degree of least squares fit of ϵ

Figure 9 Variation of standard deviation of elevation rate errors over an interval of least squares fit of azimuth. Values of $\dot{\epsilon}$ determined from time derivative of least squares fit of elevation.

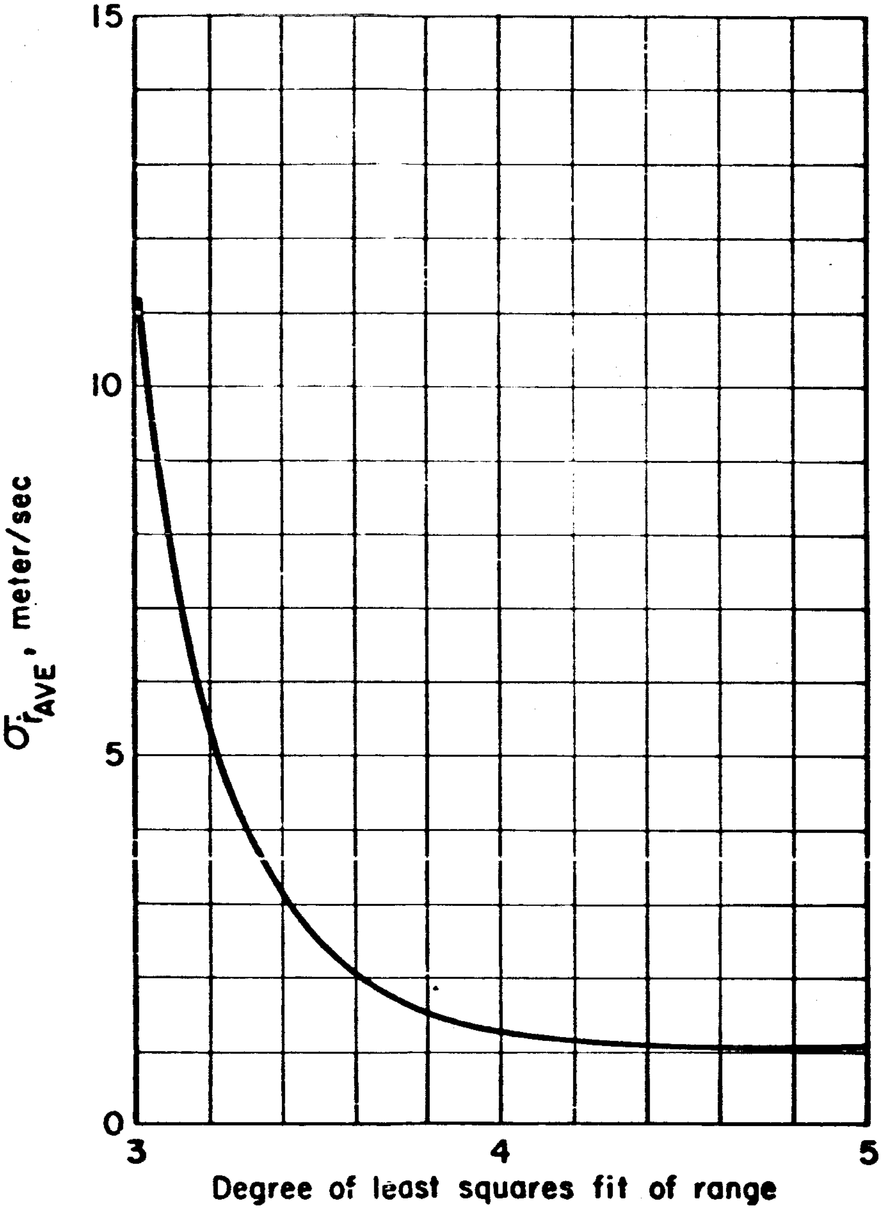


Figure 10 Average standard deviation of range rate errors vs degree of least squares fit of range.

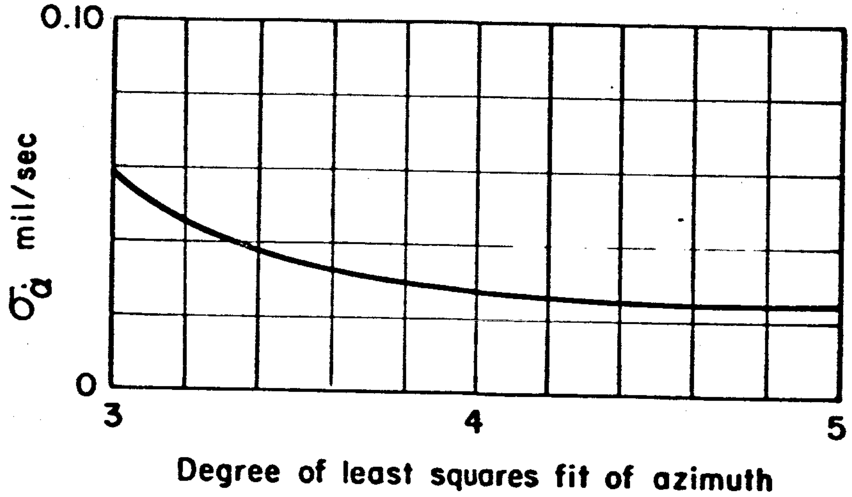


Figure 11 Average standard deviation of azimuth rate errors vs degree of least squares fit of azimuth.

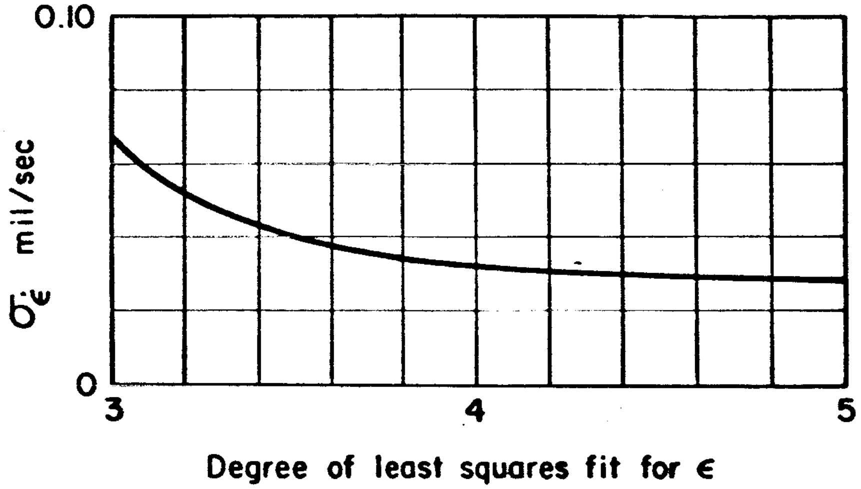


Figure 12 Average standard deviation of elevation rate errors vs degree of least squares fit of elevation.

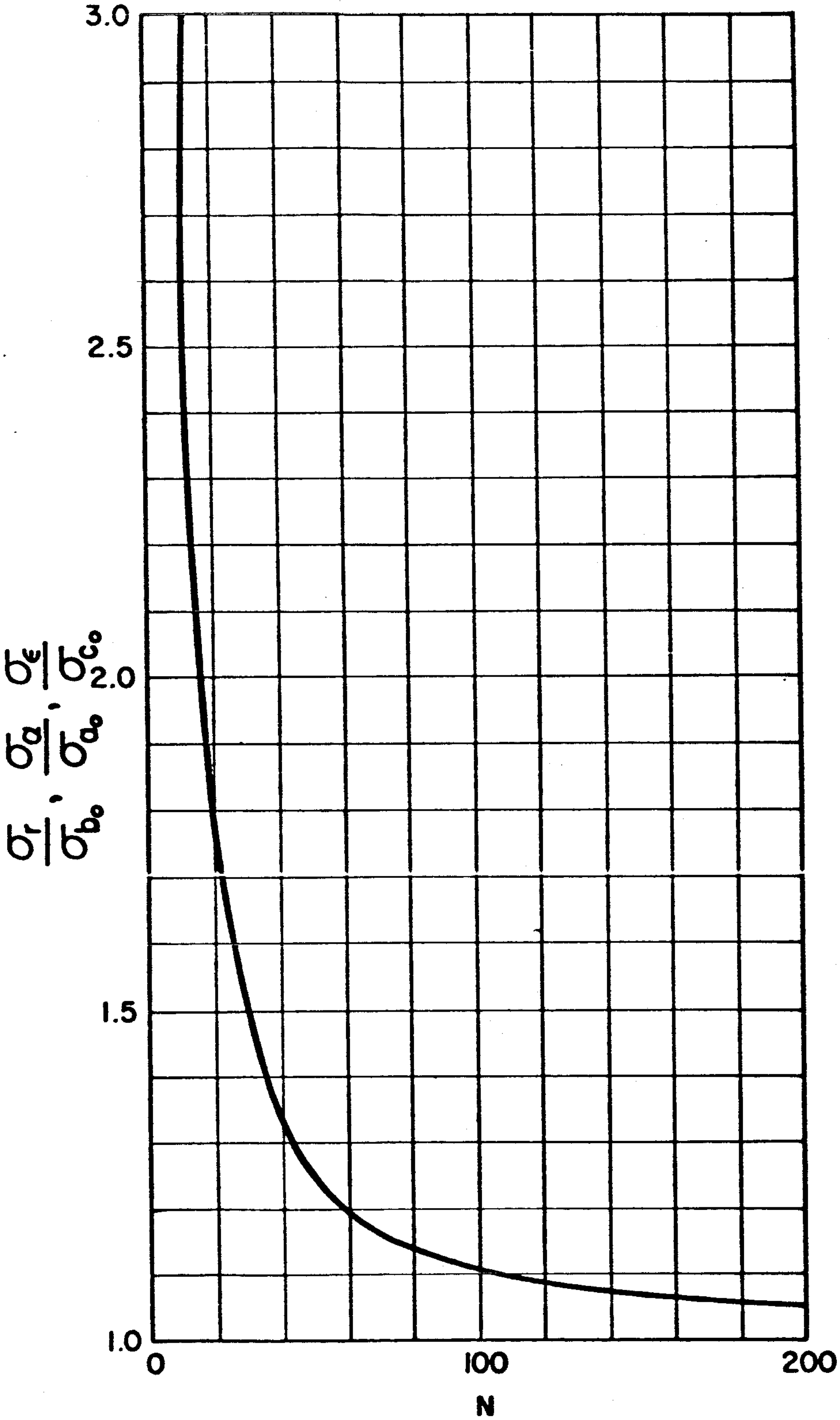


Figure 13 Effect of number of samples (N) used to determine fourth degree least squares fit of range, azimuth, elevation on expected average normalized σ_r, σ_a, σ_ϵ.

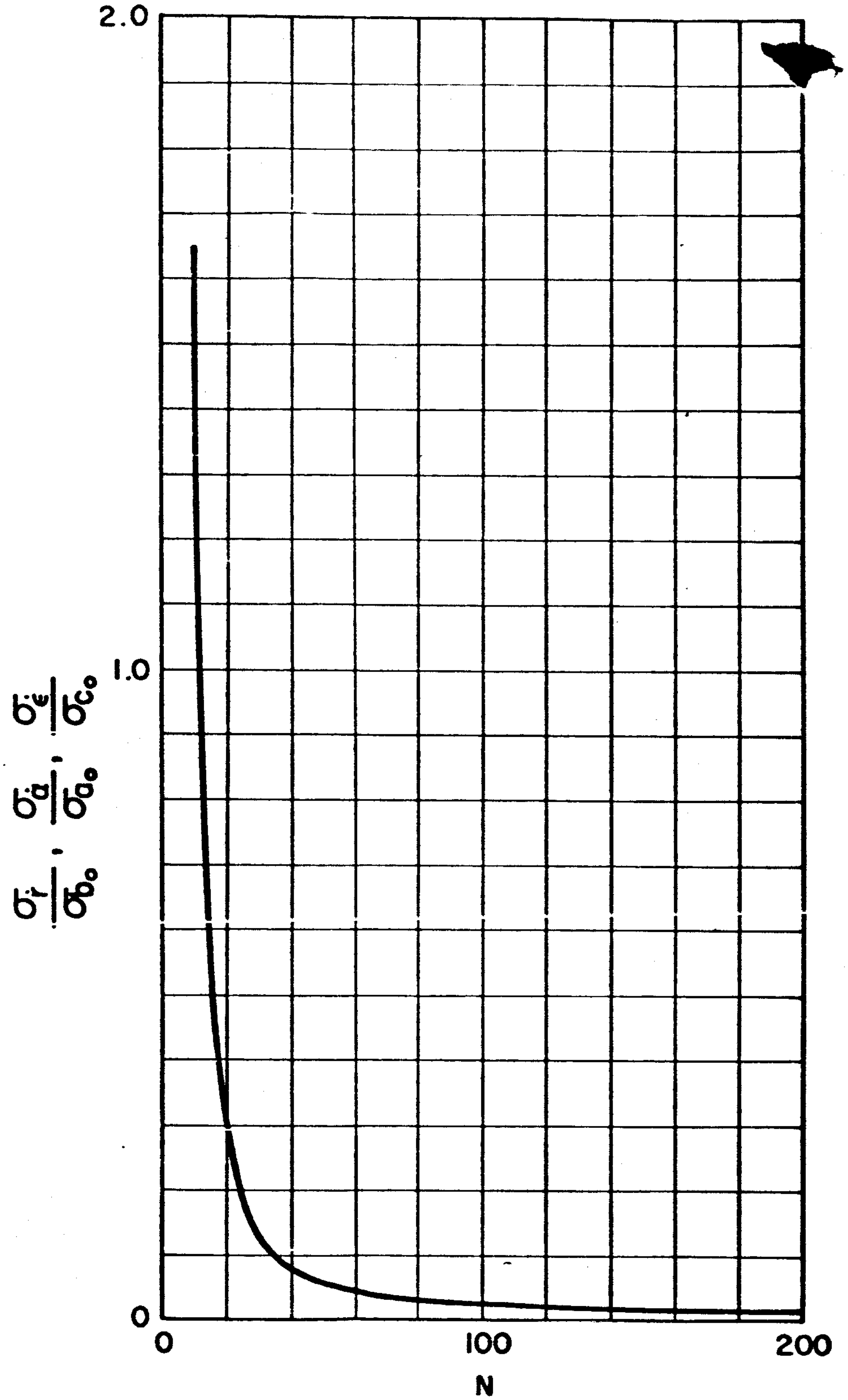


Figure 14 Effect of number of samples (N) used to determine fourth degree least squares fit of range, azimuth elevation on expected average normalized standard deviations of range rate errors, azimuth rate errors, and elevation rate errors.